THE FUNDAMENTAL THEOREM OF ALGEBRA FOR MONOSPLINES SATISFYING BOUNDARY CONDITIONS*

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ABSTRACT

The existence and uniqueness of monosplines satisfying certain boundary conditions with a maximal prescribed number of zeros is established. This result is of value in characterizing optimal quadrature formulas and in problems of best approximations involving free knots.

0. Introduction

A polynomial spline $s(x)$ of degree $n-1$ with r knots $\{\xi_i\}_1^r$ $(-\infty<\xi_1<\xi_2<\cdots<\xi_r<\infty)$ is a function of continuity class $C^{n-2}(-\infty,+\infty)$ such that $s(x)$ reduces to a polynomial of degree $n-1$ in each of the intervals $(-\infty,\xi_1)$, $[\xi_1,\xi_2),\cdots,[\xi_r,\infty)$. This concept was first formalized by Schoenberg [12] in 1946. In 1958 Schoenberg [16] introduced the notion of a monospline of degree n with k knots formed by adding the monomial x^n to a spline function of degree $n-1$ with r knots.

Monosplines arise naturally in characterizing optimal quadrature formulas for certain functions, see [13], [14] and [4]. Schoenberg [16] announced a version of the fundamental theorem of algebra for monosplines. This roughly asserts that every monospline of degree n with k knots admits at most $n+2r$ zeros and conversely, given $-\infty<t_1\leqq t_2\leqq\cdots\leqq t_{n+2r}<\infty$ obeying the restriction that

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no t_i has multiplicity exceeding $n+1$, then there exists a unique monospline of the requisite type whose zero set coincides with $\{t_1\}_1^{n+2r}$. Schoenberg proposed a proof using moment methods which appears to work only for some special cases. Karlin and Schumaker [8] provided a complete proof based on perturbations arguments, facts pertaining to totally positive transformations and the study of certain determinants. The fundamental theorem of algebra for monosplines serves a variety of applications including the problem of characterizing best approximations to certain function in the sup norm (see Johnson [3], Schumaker [17] and Fitzgerald and Schumaker [1]).

Motivated by work of Schoenberg [12] on characterizing optimal quadrature formulas ("optimality" as distinguished from "best in the sense of Sard" allows the knots in addition to the coefficients of the quadrature expression to be regarded as free variables) the first author was led to the problem of investigating the validity of the fundamental theorem of algebra for monosplines vanishing at prescribed points and also obeying suitable boundary constraints. The results were announced in Karlin [5]. The present collaboration elaborates the complete proofs of these results embracing a number of simplifications of the original arguments of the first author.

Before we proceed further it is useful to fix some notation and terminology. Let $A = \| A_{ij} \|$ then define

$$A\begin{pmatrix} i_1, \cdots, i_p \\ j_1, \cdots, j_p \end{pmatrix} = \begin{vmatrix} A_{i_1 j_1}, \cdots, A_{i_1 j_n} \\ \vdots \\ A_{i_p j_1}, \cdots, A_{i_n j_p} \end{vmatrix}$$

A is said to be sign consistent of order p (abbreviated SC_p) provided all $p \times p$ subdeterminants of A maintain a single sign; i.e., there exists $\varepsilon_p = \pm 1$ such that

$$\varepsilon_p A \begin{pmatrix} i_1, \cdots, i_p \\ j_1, \cdots, j_p \end{pmatrix} \geqq 0,$$

for all i's and j's, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_p$.

Monosplines of order n with r knots $\{\xi_i\}_{i=1}^r$, $\xi_1 < \cdots < \xi_r$ in $(0,1)$ admit the explicit representation

$$M(x) = x^n + \sum_{i=0}^{n-1} a_i x^i + \sum_{i=1}^{r} c_i (x-\xi_i)_+^{n-1} \tag{0.1}$$

$$(x_+^n = x^n \quad \text{for } x \geqq 0 \quad \text{and 0 otherwise}).$$

We will be interested in monosplines satisfying boundary conditions of the form

$$(0.2)\qquad \begin{aligned} &\mathscr{A}_k\colon \sum_{\mu=0}^{n-1} A_{\nu\mu} M^{(\mu)}(0) = 0, \qquad \nu = 1,2,\cdots,k \\ &\mathscr{B}_l\colon \sum_{\mu=0}^{n-1} B_{\lambda\mu} M^{(\mu)}(1) = 0, \qquad \lambda = 1,2,\cdots,l \end{aligned}$$

where the matrices A and B obey Postulate I below. The collection of all such monosplines will be denoted by $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$.

We stipulate to prevail throughout the sequel.

Postulate I.

(i) $0 \leqq k, l \leqq n$.

(ii) *The* $k \times n$ *matrix* $\tilde{A} = \| A_{\nu\mu}(-1)^{\mu} \|$ *is sign consistent of order* k (SC_k) *and has rank* k. *The* $l \times n$ *matrix* $B = \| B_{\nu\mu} \|$ *is* SC_l *and of rank* l.

Postulate I has wide scope. In fact, the usual types of boundary constraints occuring in the study of vibrating systems of coupled particles obey Postulate I (see Neumark [11], Gantmacher and Krein [2] and Karlin [7, Chap. 10].

Subject to a further meshing requirement on the boundary conditions (which is decribed later), we will establish the fundamental theorem of algebra for monosplines of class $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$. The precise statement is as follows.

Theorem 0.1: *If* M *is in* $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ *then* M *has at most* $n + 2r - k - l$ *zeros in* $(0,1)$. *Conversely, if* $n + 2r - k - l$ *points* $0 < t_1 \leqq t_2 \leqq \cdots \leqq t_{n+2r-k-l} < 1$ *are prescribed obeying the restriction that no* t_i *exhibits multiplicity exceeding* $n+1$, *then there exists a unique monospline* M *in* $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ *whose zeros are exactly the set of points* $\{t_i\}_1^{n+2r-k-l}$.

The result also holds for Tchebycheffian monosplines but we will confine the exposition to the case of polynomial monosplines. The extension is by now standard (e.g., see Karlin and Studden [9]).

Some general comments on the proof which is somewhat intricate. The method of Karlin and Schumaker [8] needs refinement and modification to take account of the imposed boundary conditions. We employ a continuity method which decisively relies on the implicit function theorem. The result on the fundamental theorem of algebra for monosplines of [8] serves as the starting point of the continuity method.

The proof divides into two main parts. The first part deals with the case when $l = 0$; that is, no boundary conditions are imposed at the point 1. The result from this step serves as the point of departure of an induction on the form of the boundary conditions at 1. Several interesting ancillary results emerge from our analysis.

We close this introductory section by indicating the organization of the paper. Section 1 reviews several basic properties of the fundamental spline kernel. Preliminary aspects of Rolle's theorem and interpretation of the notion of multiplicity of a zero for spline polynomials are recorded here.

Bounds on the number of zeros and their consequences is the main topic of Section 2. Relations locating the knots relative to the zeros for monosplines with a maximum number of zeros are also developed.

Section 3 is devoted to deducing a-priori bounds on the coefficients of a monospline having a full set of zeros.

The fundamental theorem of algebra for monosplines with one-sided boundary condition is established in Section 5. The proof of the general assertion of Theorem 0.1 is the content of Section 6. Some applications and extensions are noted in Section 7.

1. Preliminaries

Fundamental to the study of interpolation and approximation by splines on the interval $[0,1]$ and the development of Theorem 0.1 is the kernel function $K(z,w)$ defined on $Z \times W$, where Z and W are respectively the specific ordered sets (consisting of a set of integers and points of an open interval)

$$Z = \{x, 0, 1, \cdots, n-1;\, x \in (0,1)\}$$

and

$$W = \{0, 1, 2, \cdots, n-1, \xi;\, \xi \in (0,1)\}.$$

$K(z,w)$ is defined explicity as follows:

$$\begin{aligned}
K(x,i) &= u_i(x) = x^i, \\
K(x,\xi) &= (x-\xi)_+^{n-1}, \\
K(j,\xi) &= D_x^j \Phi(x,\xi)\big|_{x=1} (\Phi(x,\xi) = (x-\xi)_+^{n-1},\, D = \frac{d}{dx},\, D^j = D^{j-1}D), \\
K(j,i) &= D^j u_i(x)\big|_{x=1}
\end{aligned} \tag{1.1}$$

Note that in the domain Z the integers are arranged to follow the x values, while in W they are placed prior to the ξ values. The kernel $K(z,w)$ has total positivity properties important for the proof of Theorem 0.1. For other applications to the theory of interpolation of arbitrary data by splines with prescribed knots we direct the reader to [6]. For ready reference we record the result of Theorem 1.1 below. The proof can be found in [6]. Let

$$\begin{gathered}0 < x_1 \leqq x_2 \leqq \cdots \leqq x_\lambda < 1\\ 0 < \xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_\tau < 1\\ 0 \leqq i_1 < i_2 < \cdots < i_\sigma \leqq n-1\\ 0 \leqq j_1 < j_2 < \cdots < j_\rho \leqq m-1\end{gathered}$$

be arbitrary apart from the restrictions $\lambda + \rho = \sigma + \tau$;

(α) no more than n consecutive x's or ξ's coincide; and

(β) at most $n+1$ of the x's and ξ's are equal to a common value. Define

$$K\begin{pmatrix}x_1,\cdots,x_\lambda,\ j_1,\cdots,j_\rho\\ i_1,\cdots,i_\sigma,\ \xi_1,\xi_2,\cdots,\xi_\tau\end{pmatrix}$$

$$= \begin{vmatrix} u_{i_1}(x_1) & \cdots & u_{i_\sigma}(x_1) & (x_1-\xi_1)_+^{n-1} & \cdots & (x_1-\xi_r)_+^{n-1}\\ \cdot & & \cdot & \cdot & & \cdot\\ \cdot & & \cdot & \cdot & & \cdot\\ \cdot & & \cdot & \cdot & & \cdot\\ u_{i_1}(x_\lambda) & \cdots & u_{i_\sigma}(x_\lambda) & (x_\lambda-\xi_1)_+^{n-1} & \cdots & (x_\lambda-\xi_r)_+^{n-1}\\ u_{i_1}^{(j_1)}(1) & \cdots & u_{i_\sigma}^{(j_1)}(1) & u_{n-1}^{(j_1)}(1-\xi_1) & \cdots & u_{n-1}^{(j_1)}(1-\xi_r)\\ \cdot & & \cdot & \cdot & & \cdot\\ \cdot & & \cdot & \cdot & & \cdot\\ \cdot & & \cdot & \cdot & & \cdot\\ u_{i_1}^{(j_\rho)}(1) & \cdots & u_{i_\sigma}^{(j_\rho)}(1) & u_{n-1}^{(j_\rho)}(1-\xi_1) & \cdots & u_{n-1}^{(j_\rho)}(1-\xi_r) \end{vmatrix}$$

When several of the x's and/or ξ's coalesce we invoke the usual convention of replacing rows and/or columns by consecutive derivatives. The following is Theorem 1′ of [6].

LEMMA 1.1. *The kernel* $K(x,\xi)$ *is totally positive, i.e.*

$$K\begin{pmatrix}x_1,\cdots,x_\lambda,\ j_1,\cdots,j_\rho\\ i_1,\cdots,i_\sigma,\ \xi_1,\xi_2,\cdots,\xi_\tau\end{pmatrix} \geqq 0 \tag{1.2}$$

subject to (α) *and* (β) (*when n consecutive* x's (ξ's) *agree, the $n-1$th derivative in* (1.2) *is taken as a right* (*left*) *derivative*).

Moreover strict inequality occurs in (1.2) *iff when*
(a) $\sigma \geqq \lambda$ *then*

$$(1.3) \qquad \begin{aligned} &i_\mu \leqq j_{\lambda+\mu}, &&\mu = 1,2,\cdots,\sigma-\lambda \\ &x_\nu < \xi_{n-\sigma+\nu}, &&\nu = 1,2,\cdots,\lambda \end{aligned}$$

prevails, and when
(b) $\sigma < \lambda$ *then*

$$(1.4) \qquad \begin{aligned} &x_\nu < \xi_{n-\sigma+\nu}, &&\nu = 1,\cdots,\lambda \\ &\xi_\mu < x_{\sigma+\mu}\ , &&\mu = 1,2,\cdots,\lambda-\sigma \end{aligned}$$

holds with two added exceptions.

If $\tau \geqq n$ *and* $\xi_{\nu+1} = \xi_{\nu+2} = \cdots = \xi_{\nu+n}$ *for some* ν *with* $\nu + n \leqq \tau$, *then* (1.2) *is also positive where* $\xi_{\nu+1} = x_{\sigma+\nu}$. *If* $\lambda \geqq n$ *and* $x_{\mu+1} = x_{\mu+2} = \cdots = x_{\mu+n}$ *for some* μ *with* $\mu + n \leqq \lambda$, *then* (1.2) *is also strictly positive if* $x_{\mu+n} = \xi_{\mu+n-\sigma}$.

Some further notation and terminology is now appended. Let $S^-(a_1,\cdots,a_m)$ denote the number of sign changes in the sequence $a_1,\cdots,a_m$ where zero terms are discarded. Also we denote by $S^+(a_1,\cdots,a_m)$ the maximum number of sign changes achieved in the vector $(a_1,\cdots,a_m)$ by allowing each zero to be replaced by ± 1. We will need the elementary fact

$$(1.5) \qquad S^+(a_1,a_2,\cdots,a_m) + S^+(a_1,(-1)a_2,\cdots,(-1)^{m-1}a_m) \geqq m-1$$

The following lemma will be useful.

LEMMA 1.2. *Let* $f \in C^{(n)}[0,\delta]$, $\delta > 0$ *and suppose* $f^{(n)}(0) \neq 0$, *then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$

$$S^+(f(0),\ -f(0),\cdots,(-1)^n f^{(n)}(0)) = S^-(f(\varepsilon),\ -f'(\varepsilon),\cdots,(-1)^n f^{(n)}(\varepsilon)),$$

$$f^{(i)}(\varepsilon) \neq 0, \quad i = 0,1,\cdots,n \text{ and } \operatorname{sign} f^{(n)}(0) = \operatorname{sign} f^{(n)}(\varepsilon).$$

The proof involves a simple induction on n.

Let us now make precise the notion of a multiple zero of a monospline M in $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$. Since M is globally of class $C^{n-2}(-\infty,+\infty)$ a zero of order $t \leqq n-2$ has the usual interpretation

$$M(z) = \cdots = M^{(t-1)}(z) = 0 \text{ and } M^{(t)}(z) \neq 0.$$

Moreover, the multiplicity of a zero of any order is unambiguously defined provided z is distinct from a knot. In the case when ξ is a knot of M and

$$M(\xi) = \cdots = M^{(n-2)}(\xi) = 0$$

then we adopt the following convention. Set $A = M^{(n-1)}(\xi^-) = \lim_{\rho\uparrow\xi} M^{(n-1)}(\rho)$, $B = M^{(n-1)}(\xi^+) = \lim_{\rho\downarrow\xi} M^{(n-1)}(\rho)$, then ξ is said to have a zero of order

(i) $n-1$, if $A \cdot B > 0$

(ii) n, if $A \cdot B < 0$

(iii) a) n, if $A \cdot B = 0$ and $B - A > 0$

b) $n+1$, if $A \cdot B = 0$ and $B - A < 0$.

We denote by $Z(f;(0,1))$ the number of zeros of f on $(0,1)$ where a zero is counted with its multiplicity. ($Z(f;I)$ will likewise represent the number of zeros on I.) Rolle's theorem in its simplest form does not yield useful bounds for $Z(M;(0,1))$ when $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$. The following classical extension of Rolle's theorem will serve in our analysis.

Lemma 1.3. *Suppose $f \in C^{(n)}[0,1]$ and $f, f', \cdots, f^{(n)}$ have a finite number of zeros in $[0,1]$ then*

$$\begin{aligned} Z(f;(0,1)) \leqq\ & n + Z(f^{(n)};(0,1)) \\ & - S^+(f(0), -f'(0), \cdots, (-1)^n f^{(n)}(0)) \\ & - S^+(f(1)f, \cdots, f^{(n)}(1)) \end{aligned}$$

provided $f^{(n)}(0)f^{(n)}(1) \neq 0$.

For the case when f is a polynomial of exact degree n, where S^+ is replaced by S^-; this result is attributed to Fourier and Budan. A proof can be found in [7, Chapter 6]. For completeness, we sketch the main steps. An easy induction on n establishes.

$$\begin{aligned} Z(f;(\varepsilon,1-\varepsilon)) \leqq\ & n + Z(f^{(n)};(\varepsilon,1-\varepsilon)) \\ & - S^-(f(\varepsilon), -f'(\varepsilon), \cdots, (-1)^n f^{(n)}(\varepsilon)) \\ & - S^-(f(1-\varepsilon), f'(1-\varepsilon), \cdots, f^{(n)}(1-\varepsilon)) \end{aligned}$$

provided $\prod_{i=0}^n f^{(i)}(\varepsilon) f^{(i)}(1-\varepsilon) \neq 0$. Appeal to Lemma 1.2 and letting $\varepsilon \downarrow 0$ produces the desired result.

2. Bounds on $Z(M;(0,1))$

This section is devoted to developing bounds on the number of zeros of the monospline M of type $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$. A series of consequences including information relating the location of the zeros relative to the knots will also be pisclosed.

The next proposition of independent interest also has utility in the general analysis of securing bounds on $Z(M;(0,1))$.

PROPOSITION 2.1. *Let M be a monospline of degree n*

$$M(x)=x^n+\sum_{i=0}^{n-1}\lambda_i x^i+\sum_{i=1}^{r}c_i(x-\xi_i)_+^{n-1}$$

$$0<\xi_1<\xi_2<\cdots<\xi_r<1$$

then

$$Z(M;(0,1))\leqq n+2r-S^+(M(0),-M'(0),\cdots,(-1)^nM^{(n)}(0))$$
$$-S^+(M(1),M'(1),\cdots,M^{(n)}(1)).$$

Equality holds iff for i, we have

$$S^+(M(\xi_i+),-M'(\xi_i+),\cdots,(-1)^nM^{(n)}(\xi_i+))$$
$$+S^+(M(\xi_i-),M'(\xi_i-),\cdots,M^{(n)}(\xi_i-)))=n+r_i-2,$$
$$i=1,\cdots,r.$$

where r_i is the multiplicity of the zero of M at ξ_i.

PROOF. Repeated application of Lemma 1.3 yields

$$Z(M;(0,\xi_1))\leqq n-S^+(M(0),\cdots,(-1)^nM^{(n)}(0))$$
$$-S^+(M(\xi_1^-),\cdots,M^{(n)}(\xi_1^-))$$
$$Z(M;(\xi_1,\xi_2))\leqq n-S^+(M(\xi_1^+),\cdots,(-1)^nM^{(n)}(\xi_1^+))$$
$$-S^+(M(\xi_2^-),\cdots,M^{(n)}(\xi_2^-))$$

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$$Z(M;(\xi_r,1))\leqq n-S^+(M(\xi_r^+),\cdots,(-1)^nM^{(n)}(\xi_r^+))$$
$$-S^+(M(1),\cdots,M^{(n)}(1)).$$

Adding these inequalities with due account of the possible zeros at the knots produces the bound

$$Z(M;(0,1))\leqq n+2r-\sum_{i=1}^{r}W(M,\xi_i)-S^+(M(0),(-1)M'(0),\cdots,(-1)^nM^{(n)}(0))$$
$$-S^+(M(1),M'(1),\cdots,M^{(n)}(1))$$

where $W(M,\xi_i)$ is an abbreviation of the quantity

$$S^+(M(\xi_i^+), -M'(\xi_i^+), \cdots, (-1)^n M^{(n)}(\xi_i^+)) + S^+(M(\xi_i^-), M'(\xi_i^-), \cdots, M^{(n)}(\xi_i^-)) + 2 - n - r_i$$

and r_i denotes the multiplicity of the zero at ξ_i. The proof will be completed by validating the inequalities $W(M, \xi_i) \geqq 0$. This is accomplished by examination of several cases.

(i) $0 \leqq r_i \leqq n - 2$

$$\begin{aligned} W(M, \xi_i) &\geqq 2r_i + S^+(M^{(r_i)}(\xi_i^-), \cdots, M^{(n)}(\xi_i^-)) \\ &\quad + S^+((-1)^{r_i} M^{(r_i)}(\xi_i^+), \cdots, (-1)^{(n)} M^{(n)}(\xi_i^+)) + 2 - n - r_i \\ &\geqq 2 - n + r_i + S^+(M^{(r_i)}(\xi_i), \cdots, (-1)^{n-2} M^{(n-2)}(\xi_i)) \\ &\quad + S^+(M^{(r_i)}(\xi_i), \cdots, M^{(n-2)}(\xi_i)) \\ &\geqq 2 - n + r_i + n - r_i - 2 = 0 \end{aligned}$$

where the last inequality emerges by virtue of (1.5).

(ii) $r_i = n - 1$

$$W(M, \xi_i) = 1 + S^+(M^{(n-1)}(\xi_i^-), 1) + S^+(-M^{(n-1)}(\xi_i^+), 1) = 2.$$

(iii) $r_i = n$ and $M^{(n-1)}(\xi_i^-) M^{(n-1)}(\xi_i^+) < 0$.

(It is convenient at this point to refer back to the definition of multiplicity of a zero at a knot, see Section 1).

$$W(M, \xi_i) = S^+(M^{(n-1)}(\xi_i^-), 1) + S^+(-M^{(n-1)}(\xi_i^+), 1) \geqq 0.$$

(iv) $r_i = n$, $M^{(n-1)}(\xi_i^-) M^{(n-1)}(\xi_i^+) = 0$, $M^{(n-1)}(\xi_i^-) < M^{(n-1)}(\xi_i^+)$.

It follows that $W(M, \xi_i) = 2$

(v) $r_i = n + 1$ implies $W(M, \xi_i) = 0$

The proof of Proposition 2.1 is hereby complete. ‖

For M to satisfy the boundary conditions $\mathscr{A}_k$ and/or $\mathscr{B}_l$ entails certain sign change properties. The next lemma connects these notions.

LEMMA 2.1. *If* $A = \| A_{\mu\nu} \|$ *is a* $k \times n$ SC_k *matrix* ($k \leqq n$) *of rank* k *then* $Ae = 0$ *for a vector* e *requires* $S^+(e) \geqq k$.

The proof involves a direct application of Theorem 2.2 of [7, Chap. 5].

Proposition 2.1 can be combined with Lemma 2.1 to yield the desired bound on $Z(M; (0,1))$.

PROPOSITION 2.2. *Let* $M \in \mathcal{M}_{n,r}(\mathcal{A}_k \cap \mathcal{B}_l)$ *then*

$$Z(M;(0,1)) \leqq n + 2r - l - k.$$

Careful scrutiny of the proof of Proposition 2.1 reveals that when $W(M,\xi_i) = 0$ either

$$M^{(n-1)}(\xi_i^-) > 0, \; M^{(n-1)}(\xi_i^+) < 0$$

or M has a zero of order $n+1$ at ξ_i and then

$$M^{(n-1)}(\xi_i^+) - M^{(n-1)}(\xi_i^-) < 0.$$

Stating formally this fact, we have

PROPOSITION 2.3. *Let M be a monospline of degree n with r knots interior to* $[0,1]$, *viz*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}.$$

If

$$Z(M;(0,1)) = n + 2r - S^+(M(0), -M'(0), \cdots, (-1)^n M^{(n)}(0)) - S^+(M(1), M'(1), \cdots, M^{(n)}(1))$$

then

$$c_i < 0, \qquad i = 1, 2, \cdots, r.$$

The next proposition concerns bounds on zeros of derivatives $M^{(i)}(x)$ of $M(x)$.

PROPOSITION 2.4. *Let*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}$$

$$0 < \xi_1 < \cdots < \xi_r < 1.$$

If

$$Z(M;(0,1)) = n + 2r - S^+(M(0), -M'(0), \cdots, (-1)^n M^{(n)}(0)) - S^+(M(1), M'(1), \cdots, M^{(n)}(1))$$

then

$$Z(M';(0,1)) = n - 1 + 2r - S^+(M'(0), \cdots, (-1)^{n-1} M^{(n)}(0)) - S^+(M'(1)) \cdots, M^{(n)}(1)). \tag{2.1}$$

Moreover,

$$Z(M^{(n-1)};\ (-\infty, +\infty)) = 2r + 1.$$

PROOF. Note with the aid of Lemma 1.2 we may assume without restricting generality that $M^{(i)}(0)M^{(i)}(1) \neq 0$, $i = 0, 1, \cdots, n$. Subject to the zero convention set forth in Section 1, it is straightforward to strengthen the conclusion of Rolle's theorem asserting specifically that

$$Z(M;(0,1)) + S^-(M(0), -M'(0)) + S^-(M(1), M'(1)) \leqq Z(M';(0,1)) + 1.$$

This inequality in conjunction with the hypothesis implies

$$\begin{aligned} Z(M';(0,1)) &\geqq n - 1 + 2r - S^-(M(0),\cdots,(-1)^n M^{(n)}(0)) - S^-(M(1),\cdots,M^{(n)}(1)) \\ &\qquad + S^-(M(0), -M'(0)) + S^-(M(1), M'(1)) \\ &= n - 1 + 2r - S^-(M'(0),\cdots,(-1)^{n-1}M^{(n)}(0)) \\ &\qquad - S^-(M'(1),\cdots,M^{(n)}(1)). \end{aligned}$$

Proposition 2.1 applied to M' combined with the inequality just proved confirms relation (2.1). Repeated application now gives

$$Z(M^{(n-1)};(0,1)) = 2r + 1 - S^-(-M^{(n-1)}(0), 1) - S^-(M^{(n-1)}(1), 1).$$

Since

$$M^{(n-1)}(x) = \begin{cases} x + \alpha & x \leqq 0 \\ x + \beta & x \geqq 1 \end{cases}$$

it is easy to check that

$$\begin{aligned} Z(M^{(n-1)};(-\infty,0]) &= S^-(-M^{(n-1)}(0),1) \\ Z(M^{(n-1)};\ [1,\infty)) &= S^-(M^{(n-1)}(1),1). \end{aligned}$$

PROPOSITION 2.5. *Suppose* $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ *and*

$$Z(M;(0,1)) = n + 2r - k - l$$

then

(i) $S^+(M(0),\cdots,(-1)^{n-1}M^{(n-1)}(0)) = S^+(M(0),\cdots,(-1)^n M^{(n)}(0)) = k$

(ii) $S^+(M(1),\cdots,M^{(n-1)}(1)) = S^+(M(1),\cdots,M^{(n)}(1)) = l$ (2.2)

(iii) $Z(M^{(n-1)};[0,1]) = 2r + 1$.

Proof. By Lemma 2.1

$$S^+(M(0),\cdots,(-1)^{n-1}M^{(n-1)}(0)) \geqq k$$

$$S^+(M(1),\cdots,M^{(n-1)}(1)) \geqq l$$

and so

$$k + l \leqq S^+(M(0),\cdots,(-1)^n M^{(n)}(0)) + S^+(M(1),\cdots,M^{(n)}(1)).$$

Comparing these inequalities with the conclusion of Proposition 2.1 we see that statements (i) and (ii) are correct. It follows that

$$M^{(n-1)}(1) \geqq 0, \; M^{(n-1)}(0) \leqq 0.$$

Hence, from Proposition 2.4, we infer that

$$Z(M^{(n-1)}; [0,1]) = 2r + 1. \qquad \|$$

Consider boundary conditions of the form

$$\begin{aligned} &M^{(i_\mu)}(0) = 0, \qquad \mu = 1,\cdots,k \\ (2.3) \qquad &M^{(j_\nu)}(1) = 0, \qquad \nu = 1,\cdots,l \end{aligned}$$

$$0 \leqq i_1 < \cdots < i_k \leqq n-1, \; 0 \leqq j_1 < \cdots < j_l \leqq n-1.$$

As a consequence of (2.2) we deduce

PROPOSITION 2.6. *Let M be a monospline of degree n with r knots in* $(0,1)$

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x-\xi_i)_+^{n-1}.$$

If $Z(M;(0,1)) = n + 2r - k - l$ *and M satisfies* (2.3), *then*

(a) $\qquad (-1)^\nu M^{(i)}(1) > 0, \qquad i_{k-\nu} < i < i_{k-\nu+1}, \qquad \nu = 0,1,\cdots,k$

(b) $\qquad (-1)^\nu(-1)^j M^{(j)}(0) > 0, \; j_{l-\nu} < j < j_{l-\nu+1}, \qquad \nu = 0,1,\cdots,l$

(*where* $i_0 = j_0 = -1, \; i_{k+1} = j_{l+1} = n$) .

We continue in the next proposition by locating the zeros relative to the knots when the monospline at hand exhibits a full set of real zeros. Set

$$p = S^+(M(0),\cdots,(-1)^n M^{(n)}(0))$$

$$q = S^+(M(1),\cdots,M^{(n)}(1)).$$

PROPOSITION 2.7. *Let M be a monospline of the form*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x-\xi_i)_+^{n-1}, \qquad 0 < \xi_1 < \cdots < \xi_r < 1.$$

If $n + 2r - p - q$ distinct points exist satisfying

$$M(t_i) = 0, \qquad i = 1, \cdots, n + 2r - p - q$$

$$0 < t_1 < \cdots < t_{n+2r-p-q} < 1$$

then setting $\xi_i^* = \xi_{[(i+1)/2]}$, $i = 1, \cdots, 2r$, $\left(\left[\frac{i+1}{2}\right]\right.$ *denotes the greatest integer* $\left.\leqq \frac{i+1}{2}\right)$ *we have*

$$t_\nu < \xi_{p+\nu}^* \tag{2.4}$$

$$\xi_\nu^* < t_{n-q+\nu} \tag{2.5}$$

whenever the indices make sense and $n > 1$.

PROOF OF (2.4). Suppose to the contrary that $\xi_{p+\nu}^* \leqq t_\nu$ for some ν. Then

$$\hat{M}(x) = x^n + \sum_{i=0}^{n-1} \hat{\lambda}_i x^i + \sum_{[(p+\nu)/2]+1}^{r} c_i (x - \xi_i)_+^{n-1}$$

coincides with M on $[\xi_{p+\nu}^*, 1]$ for suitable $\hat{\lambda}_i$. According to Proposition 2.1

$$Z(\hat{M}; [\xi_{p+\nu}^*, 1)) \leqq n + 2\left\{r - \left[\frac{p+\nu}{2}\right]\right\} - q.$$

But, manifestly

$$Z(\hat{M}; [\xi_{p+\nu}^*, 1)) = Z(M; [\xi_{p+\nu}^*, 1)) \geqq n + 2r - p - q - (\nu - 1)$$

yielding $2[(p + \nu)/2] \leqq p + \nu - 1$, an absurdity. The relations (2.5) are proved analogously.

REMARK. In the case $n = 1$ it can be checked that

$$\xi_i = t_{2i-p} \quad (2i > p).$$

Furthermore, provided M is continuous, Proposition 2.7 persists even when the t_i's are multiple zeros.

This section is concluded by adding a postulate pertaining to the meshing of the boundary conditions at points 0 and 1. The stipulations at the end points and the number of knots cannot be prescribed completely independently and be concordant with Theorem 0.1.

POSTULATE II

(i) *Postulate I*

(ii) *If $l > 2r$ then there exists indices*

$$0 \leqq i_1 < \cdots < i_k \leqq n - 1, \; 0 \leqq j_1 < \cdots < j_l \leqq n - 1$$

satisfying

$$A\binom{1,\cdots,k}{i_1,\cdots,i_k}\neq 0,\qquad B\binom{1,\cdots,l}{j_1,\cdots,j_l}\neq 0$$

and

$$j_\mu \leqq i'_{\lambda+\mu},\quad \mu = 1,\cdots,l-2r,\quad \lambda = n+2r-k-l \tag{2.6}$$

where $\{i'_1,\cdots,i'_{n-k}\}$ *denote the complementary indices of* $\{i_1,\cdots,i_k\}$ *in the set* $\{0,1,\cdots,n-1\}$.

The relevance of Postulate II will be realized in the general analysis of Theorem 0.1. For the moment we highlight its utility. Construct the mapping of Euclidean $n+2r$ space into itself as follows

$$\Phi\colon E^{n+2r}\to E^{n+2r},$$

where for each

$$\boldsymbol{u} = (\lambda_0,\lambda_1,\cdots,\lambda_{n-1},\ \xi_1,\ c_1,\cdots,\xi_r,\ c_r)\in E^{n+2r}$$

we determine the monospline

$$M(x) = x^n + \sum_{i=0}^{n-1}\lambda_i x^i + \sum_{i=1}^{r} c_i(x-\xi_i)_+^{n-1}.$$

Given $0 < t_1 < \cdots < t_{n+2r-k-l} < 1$ form

$$(\Phi(\boldsymbol{u}))_i = \begin{cases}\displaystyle\sum_{j=0}^{n-1} A_{ij}M^{(j)}(0) & 1\leqq i\leqq k\\ M(t_{i-k}) & k< i\leqq n+2r-l\\ \displaystyle\sum_{j=0}^{n-1} B_{i-n-2r+l,j}M^{(j)}(1), & n+2r-l< i\leqq n+2r\end{cases}$$

Let $J(\boldsymbol{u})$ denote the Jacobian of Φ at $\boldsymbol{u}$. Then, reliance on Laplace's expansion and the Cauchy Binet formula (see [6] for details) produces the result

$$J(\boldsymbol{u}) = \left(\prod_{i=1}^{r} c_i\right)(-1)^{k((k-1)/2)}\sum_{\substack{0\leqq i_1<i_2\leqq\cdots<i_k\leqq n-1\\ 0\leqq j_1<j_2\cdots<j_l\leqq n-1}}$$

$$\times\,\tilde{A}\binom{1,2,\cdots,k}{i_1,i_2,\cdots,i_k}B\binom{1,\cdots,l}{j_1,\cdots,j_l}K\binom{t_1,\cdots,t_{n+2r-k-l},j_1,\cdots,j_l}{i'_1,\cdots,i'_{n-k},\ \xi_1,\xi_1,\xi_2,\xi_2,\cdots,\xi_r,\xi_r} \tag{2.7}$$

where $\tilde{A}_{\mu\nu} = A_{\mu\nu}(-1)^\nu$ and $\{i'_1,i'_2,\cdots,i'_{n-k}\}$ are the complementary indices of $\{i_1,\cdots,i_k\}$ in $\{0,1,\cdots,n-1\}$. (Pertaining to the notation and meaning of $K(:\,:\,:)$, see Section 1.) This calculation in view of Lemma 1.1, Propositions 2.3 and 2.4 implies that if the boundary conditions $\mathscr{A}_k \cap \mathscr{B}_l$ fulfill the requirements of Postulate II then

$$\Phi(\boldsymbol{u}) = 0 \text{ entails } J(\boldsymbol{u}) \neq 0. \tag{2.8}$$

This fact provides a vital ingredient in the ensuing analysis.

3. Bounds on the coefficients of monosplines

In Section 2 we determined the upper bound $n + 2r - k - l$ on the number of zeros of a monospline $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$. Several properties of monosplines with a maximum number of zeros were listed in Propositions 2.3–2.7. Our objective in this section will be to provide a-priori estimates on the coefficients of such a monospline where $\mathscr{A}_k \cap \mathscr{B}_l$ obeys Postulate II. It is proved in Karlin and Schumaker [8] that if

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1}$$

has $n + 2r$ zeros in some bounded interval I then λ_i and c_i are uniformly bounded independent of the location of the zeros in I. Can this result be extended to the circumstance of $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ exhibiting a maximum number of zeros in $I = (0,1)$ and where $\mathscr{A}_k \cap \mathscr{B}_l$ obey Postulate II? The next example indicates that we cannot expect such a result without appending some further constraints.

Example.

$$\begin{aligned} r &= 0, n = 3, k = 1, l = 1. \\ M(x) &= x^3 - (\alpha + 6)x + \alpha, \alpha > 0 \\ \mathscr{A}_1 &: M''(0) = 0 \\ \mathscr{B}_1 &: (6/5)M(1) + M''(1) = 0. \end{aligned}$$

Here, $\mathscr{A}_1 \cap \mathscr{B}_1$ satisfies Postulate II ($i_1 = 2$, $j_1 = 0$). According to Proposition 2.1, M can have at most one zero in $(0,1)$. Since $M(0)M(1) = -5\alpha < 0$, M, in fact, exhibits a zero in $(0,1)$, but the coefficients of M are not uniformly bounded.

Note that the zero of M in $(0,1)$ tends to the boundary point 1 as $\alpha \to +\infty$. This is the nub of the difficulty. We will prove that boundedness of the coefficients is maintained provided the zeros are kept away from the end points of $[0,1]$.

Boundedness of *certain* coefficients holds *with no restrictions* on the location of zeros of M in $(0,1)$. This is the principal content of Propositions 3.1 and 3.2

PROPOSITION 3.1. *Suppose $\mathscr{A}_k \cap \mathscr{B}_l$ satisfies Postulate I. Let $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}$$

and $Z(M; (0,1)) = n + 2r - k - l$. Then

$$|\lambda_{n-1}| \leqq n, \ |c_i| \leqq n, \qquad i = 1, \cdots, r.$$

PROOF. According to Proposition 2.5, we have

$$Z(M^{(n-1)}; [0,1]) = 2r + 1$$

and

$$M^{(n-1)}(x) = n!\, x + (n-1)!\, \lambda_{n-1} + \sum_{i=1}^{r} (n-1)!\, c_i (x - \xi_i)_+^0.$$

Consider for simplicity the case where $M^{(n-1)}$ displays only distinct zeros $t_1 < \cdots < t_{2n+1}$ (the case of multiple zeros works similarly but requires more tedious examination of cases). The remark following Proposition 2.7 tells us that ξ_i must be equal to t_{2i}. Hence

$$n!\, t_1 + (n-1)!\, \lambda_{n-1} = 0$$

$$t_{2i+1} n! + (n-1)!\, \lambda_{n-1} + (n-1)! \sum_{\mu=1}^{i} c_\mu = 0$$

so that

$$c_i = n[t_{2i-1} - t_{2i+1}], \ \lambda_{n-1} = -nt_1 \cdot \|$$

The previous proposition established that the coefficients λ_{n-1}, $c_1, \cdots, c_r$ are bounded independent of the elements comprising the boundary forms as long as Postulate I is satisfied. A parallel result relevant for all the coefficients holds provided $2r \geqq l$ (or $2r \geqq k$) that is, where there are sufficiently many knots compared to the number of boundary conditions at 1 (or 0). More specifically:

PROPOSITION 3.2. *Let $\mathscr{A}_k \cap \mathscr{B}_l$ satisfy Postulate I. Suppose $2r \geqq l$. Then there exists a constant K such that if*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}$$

belongs to $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ then

$$|\lambda_i| \leqq K, \qquad i = 0, \cdots, n-1$$

$$|c_i| \leqq K, \qquad i = 1, \cdots, r.$$

The bound K depends on n, r, k, l but not on the elements of the matrices $\| A_{\mu\nu} \|$ and $\| B_{\mu\nu} \|$.

PROOF. The use of Rolle's theorem coupled with induction (cf. Lemma 1.3) shows that

$$Z(M^{(j)}; [0,1]) \geqq Z(M; (0,1)) - j + v_j, \qquad j = 0,1,\cdots,n-1$$
$$v_j \equiv S^+(M(0),\cdots,(-1)^j M^{(j)}(0)).$$

But (see Proposition (2.5))

$$\begin{aligned} k &\leqq S^+(M(0),\cdots,(-1)^{n-1}M^{(n-1)}(0)) \\ &\leqq v_j + S^+(M^{(j)}(0),\cdots,(-1)^{n-1}M^{(n-}(0)) \\ &\leqq v_j + n - j - 1. \end{aligned}$$

Hence

$$Z(M^{(j)}; [0,1]) \geqq (n + 2r - k - l) - j + (k - n + j + 1) = 2r - l + 1 \geqq 1.$$

Thus

$$Z(M^{(j)}; [0,1]) \geqq 1, \qquad j = 0,1,\cdots,n-1. \tag{3.1}$$

Bounds on $\lambda_{n-2}, \lambda_{n-3},\cdots,\lambda_0$ are established inductively by using the conclusion of Proposition 3.1, coupled with the information of (3.1).

A parallel proof works if $2r \geqq k$ by transforming the variable x to $1 - x$. ‖

When $2r < l$ we have seen in Example 1 that boundedness does not necessarily prevail. However the following fact will serve our needs.

PROPOSITION 3.3. *Let $\| A_{\mu\nu} \|$, $\| B_{\mu\nu} \|$ induce boundary conditions fulfilling Postulate II. Suppose $2r < l$. Given $\delta > 0$, there exists a constant D (depending on δ, $\| A_{\mu\nu} \|$, $\| B_{\mu\nu} \|$) such that for*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}$$

in $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ satisfying $M(t_i) = 0$, $i = 1,\cdots,n + 2r - k - l$ with

$$\delta < t_1 < \cdots < t_{n+2r-k-l} < 1 - \delta$$

then it follows that

$$|\lambda_i| \leqq D, \qquad i = 0,1,\cdots,n-1$$
$$|c_i| \leqq D, \qquad i = 1,2,\cdots,r$$

PROOF. Define

$$Y(x) = x^n + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1}$$

$$P(x) = \sum_{i=0}^{n+2r-1} \lambda_i x$$

with λ_i specified equal to 0 for $i \geqq n$. Of course, $M(x) = P(x) + Y(x)$. If $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$, then Proposition 3.1 assures the existence of a constant $\bar{K}$ such that

$$\max_{0 \leqq x \leqq 1} \left| Y^{(i)}(x) \right| \leqq \bar{K}, \qquad i = 0, 1, \cdots, n-2 \tag{3.2}$$

$$\left| \sum_{\nu=0}^{n-1} A_{\mu\nu} Y^{(\nu)}(0) \right| \leqq \bar{K}, \qquad \mu = 1, \cdots, k \tag{3.3}$$

$$\left| \sum_{\nu=0}^{n-1} B_{\sigma\nu} Y^{(\nu)}(1) \right| \leqq \bar{K}, \qquad \sigma = 1, \cdots, l. \tag{3.4}$$

Set

$$H_\mu = \sum_{\nu=0}^{n-1} A_{\mu\nu} Y^{(\nu)}(0), \qquad \mu = 1, \cdots, k$$

$$L_\mu = \sum_{\nu=0}^{n-1} B_{\mu\nu} Y^{(\nu)}(1), \qquad \mu = 1, \cdots, l.$$

Abbreviate $n + 2r = N$. Since $M \in \mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ we have

$$\begin{aligned} &\sum_{\nu=0}^{n-1} A_{\mu\nu} P^{(\nu)}(0) = -H_\mu, && \mu = 1, \cdots, k \\ &P(t_i) = -Y(t_i), && i = 1, \cdots, N-k-l \\ &\sum_{\nu=0}^{n-1} B_{\mu\nu} P^{(\nu)}(1) = -L_\mu, && \mu = 1, \cdots, l. \end{aligned} \tag{3.5}$$

It is convenient to introduce the quantities

$$\hat{B}_{\mu\nu} = \sum_{\sigma=0}^{n-1} B_{\mu\sigma} \frac{d^\sigma}{dx^\sigma} \left(\frac{x^\nu}{\nu!} \right) \Bigg|_{x=1} \quad \mu = 1, \cdots, l,\ \nu = 0, 1, \cdots, N-1.$$

Regard (3.5) as a non-homogeneous system of N linear equations in the variables $\lambda_0, \lambda_1, \cdots, \lambda_{N-1}$ with determinant given by:

$$\Delta = \begin{vmatrix} A_{10}, & A_{11}, & \cdots, & A_{1,n-1}, & 0,\cdots, & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ A_{k0}, & A_{k1}, & \cdots, & A_{k,n-1}, & 0,\cdots, & 0 \\ t_1^0, & t_1^1, & \cdots, & & & t_1^{N-1} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ t_{N-k-l}^0, & t_{N-k-l}^1, & \cdots, & & & t_{N-k-l}^{N-1} \\ \hat{B}_{10}, & \hat{B}_{11}, & \cdots, & & & \hat{B}_{1,N-1} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \hat{B}_{l0}, & \hat{B}_{l1}, & \cdots, & & & \hat{B}_{l,N-1} \end{vmatrix}$$

Let $\Delta^{(\nu)}$ be the determinant obtained from Δ by substituting for the νth column the vector of components

$$(-H_1,\cdots,-H_k,-Y(t_1),\cdots,-Y(t_{N-k-l}),-L_1,\cdots,-L_l).$$

Cramér's rule gives the formula

$$\nu!\,\lambda_\nu = \frac{\Delta^{(\nu)}}{\Delta}. \tag{3.6}$$

Set

$$D(t_1,\cdots,t_{N-k-l}) = \prod_{i<j}(t_i - t_j).$$

For all

$$0 \leqq t_1 < t_2 < \cdots < t_{N-k-l} \leqq 1$$

with due account of (3.2)–(3.5) we deduce the existence of a constant E independent of $\{t_i\}$, such that

$$\left|\frac{\Delta^{(\nu)}}{D(t_1,\cdots,t_{N-k-l})}\right| \leqq E\cdot \max_{0\leqq x\leqq 1} \left|Y^{(N-k-l-1)}(x)\right|.$$

Since $2r < l$ and the boundary conditions fulfill the stipulations of Postulate II we are assured of indices

$$0 \leqq i_1 < \cdots < i_k \leqq n-1$$

$$0 \leqq j_1 < \cdots < j_l \leqq n-1$$

with the properties

$$A\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix} \neq 0$$

$$B\begin{pmatrix}1,\cdots,l\\ j_1,\cdots,j_l\end{pmatrix} \neq 0 \tag{3.8}$$

and

$$j_\mu \leqq i'_{N-k-l+\mu}, \qquad \mu = 1,\cdots,l.$$

Expanding Δ (compare with 2.7) we find

$$|\Delta| \geqq \gamma K\begin{pmatrix}t_1,\cdots,t_{N-k-l}, & j_1,\cdots,j_l\\ i'_1,\cdots,i'_{N-k-l}, & i'_{N-k-l+1},\cdots,i'_{N-k}\end{pmatrix} \tag{3.9}$$

where $\gamma > 0$ is a constant independent of $t_1 < \cdots < t_{N-k-l}$ but depending on the non-zero values in (3.8). Let

$$K(t) = K^*\begin{pmatrix}t,\cdots,t, & j_1,\cdots,j_l\\ i'_1,\cdots,i'_{N-k} & \end{pmatrix}$$

then $K(t) > 0$, $0 < t < 1$ by Lemma 1.1. But then it follows as in Chapter 2 of [7] that

$$K\begin{pmatrix}t_1,\cdots,t_{N-k-l}, & j_1,\cdots,j_l\\ i'_1,\cdots, & i'_{N-k}\end{pmatrix} \geqq \bar{\gamma} D(t_1,\cdots,t_{N-k-l}) \tag{3.10}$$

for some absolute constant $\bar{\gamma} > 0$ provided $0 < \delta \leqq t_1$ and $t_{N-k-l} \leqq 1-\delta < 1$.

Combining the estimates (3.7), (3.9) and (3.10) in (3.6) validates the bounds as stated in the Proposition.

4. Uniqueness

In this section we establish the uniqueness assertion of Theorem 0.1. Some reductions needed for the task of the proof of existence in Theorem 0.1 are also set forth.

In this section we assume $n \geqq 3$. The cases $n = 1$ and $n = 2$ will be treated separately in Section 5.

PROPOSITION 4.1. *Let*

$$M(x) = x^n + \sum_{i=0}^{n-1} b_i x^i + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1}$$

$$N(x) = x^n + \sum_{i=0}^{n-1} d_i x^i + \sum_{i=1}^{r} e_i(x - \eta_i)_+^{n-1}$$

be two monosplines sharing the set of zeros $0 < t_1 \leqq t_2 \leqq \cdots \leqq t_{n+2r-k-l} < 1$ *where no zero exhibits multiplicity exceeding* $n + 1$. *Suppose* M *and* N *belong to* $\mathcal{M}_{n,r}(\mathcal{A}_k \cap \mathcal{B}_l)$ *where boundary conditions* $\mathcal{A}_k \cap \mathcal{B}_l$ *are of the type fulfilling Postulate II. Then*

$$M \equiv N.$$

PROOF. Form

$$P \equiv M - N = \sum_{i=0}^{n-1} (b_i - d_i)x^i + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1} - \sum_{i=1}^{r} e_i(x - \eta_i)_+^{n-1}. \tag{4.1}$$

We restrict consideration to the case that the multiplicity of any zero among $\{t_i\}_1^{n+2r-k-l}$ occuring at the knots are each $\leqq n - 1$. The discussion for the case involving the presence of zeros of multiplicity $\geqq n$ is left to the reader.

Let $\zeta_1, \cdots, \zeta_{2r}$ be the sequence $\{\xi_i\} \cup \{\eta_i\}$ (where ξ_i is repeated twice if $\xi_i = \eta_i$) arranged in natural order. Then P is a spline of order n with $2r$ knots satisfying

$$\begin{aligned}
&\sum_{j=0}^{n-1} A_{ij} P^{(j)}(0) = 0, \qquad && i = 1, \cdots, k \\
&P(t_i) = 0, && i = 1, 2, \cdots, n + 2r - k - l \\
&\sum_{j=0}^{n-1} B_{ij} P^{(j)}(1) = 0, && i = 1, \cdots, l.
\end{aligned} \tag{4.2}$$

This display involves a set of $n + 2r$ linear homogeneous equations in the $n + 2r$ coefficients of $P(x)$. We will show that the determinant of this system of equations is nonzero. This fact manifestly implies $P \equiv 0$ the required uniqueness.

Expanding the determinant of the system as in (2.7) involves terms of a common sign. The non-vanishing of the determinant is equivalent to ascertaining indices $i_1 < \cdots < i_k$, $j_1 < \cdots < j_l$ such that

$$(4.3)\qquad A\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix}B\begin{pmatrix}1,\cdots,l\\ j_1,\cdots,j_l\end{pmatrix}K\begin{pmatrix}t_1,\cdots,t_{n+2r-k-l},j_1,\cdots,j_l\\ i'_1,\cdots,i'_{n-k},\zeta_1,\cdots,\zeta_{2r}\end{pmatrix}\neq 0.$$

Recalling the prescriptions of Postulate II (Section 2), and consulting Lemma 1.1, we find that the quantity in (4.3) is non-zero provided

$$(4.4)\qquad t_\nu < \zeta_{k+\nu}$$

$$(4.5)\qquad \zeta_\nu < t_{n-k+\nu}$$

hold (whenever the indices make sense).

Now Proposition 2.7 informs us that

$$(4.6)\qquad t_\nu < \xi_{[(k+\nu+1)/2]}$$

$$(4.7)\qquad \xi_{[k+\nu+1)/2]} < t_{n-k+\nu}$$

and

$$(4.8)\qquad t_\nu < \eta_{[k+\nu+1)/2]}$$

$$(4.9)\qquad \eta_{[(k+\nu+1)/2]} < t_{n-k+\nu}$$

The relations (4.6)–(4.9) assure (4.4) and (4.5). Indeed, from the definition of the ζ sequence we know that

$$\min\left[\xi_{[(i+1)/2]},\eta_{((i+1)/2]}\right] \leqq \zeta_i, \qquad i=1,\cdots,r.$$

Thus (4.4) follows from either (4.6) and/or (4.8). A similar argument emanating from (4.7) and (4.9) established (4.5). ‖

REMARK. If the boundary conditions have the form

$$\sum_{j=0}^{n-1} A_{ij}(-1)^j M^{(j)}(0)=0, \qquad i=1,\cdots,k$$

$$\sum_{j=0}^{n-1} A_{ij}M^{(j)}(1)=0, \qquad i=1,\cdots,k$$

and the zero set of M is invariant under the transformation $x\to 1-x$ then the uniqueness guarantees that the monospline satisfying Theorem 0.1 has the symmetry property

$$M(x) = \ -1)^n M(1-x).$$

This functional relationship facilitates the practical computation of the desired M.

We conclude this section by proving that the existence part of Theorem 0.1. involving zeros of higher multiplicities is a consequence of knowing the existence for the case of simple zeros. Suppose Theorem 0.1 is achieved in the case of simple zeros. Now let $0 < t_1 \leqq t_2 \leqq \cdots \leqq t_{n+2r-k-l} < 1$ be prescribed.

For each integer $e \geqq 1$, construct a set of points $\{s_i(e)\}_{i=1}^{n+2r-k-l}$ by "spreading apart" the multiple zeros. Specifically, if

$$t_{m-1} < t_m = t_{m+1} = \cdots = t_{m+p} < t_{m+p+1}$$

define $s_i(e) = t_m + j\varepsilon/2^e$, $j = 0, 1, \cdots, p$ where ε is determined so that $s_i(e)$ are distinct and in $(0,1)$; viz

$$0 < s_1(e) < \cdots < s_{n+2r-k-l}(e) < 1.$$

There exists a monospline $M_{(e)}$ in $\mathscr{M}_{n,r}(\mathscr{A}_k \cap \mathscr{B}_l)$ such that

$$M_{(e)}(x) = x^n + \sum_{i=0}^{n-1} \lambda_i^{(e)} x^i + \sum_{i=1}^{r} c_i^{(e)}(x - \xi_i^{(e)})_+^{n-1}$$

satisfies

$$\begin{aligned} &\sum_{\nu=0}^{n-1} A_{\mu\nu} M_{(e)}^{(\nu)}(0) = 0, \qquad && \mu = 1, \cdots, k \\ (4.10) \qquad &M_{(e)}(s_i(e)) = 0, && i = 1, \cdots, n + 2r - k - l \\ &\sum_{\nu=0}^{n-1} B_{\mu\nu} M_e^{(\nu)}(1) = 0, && \mu = 1, \cdots, l \end{aligned}$$

Propositions 3.1 and 3.2 affirm that the coefficients of M_e are uniformly bounded. We invoke the standard selection process to achieve a monospline

$$M(x) = \lim_{m \to \infty} M_{e_m}(x).$$

The equations (4.10) without difficulty pass into the required relations for $M(x)$.

5. Existence for one-sided boundary conditions

It is useful to outline the steps of the analysis. We already know from the discussion of Section 4 that it is enough to deal only with the case of simple zeros. The special case of $n = 1$ or 2 is easily handled by explicit construction. When $n \geqq 3$ we proceed by a continuity argument with heavy reliance on the implicit function theorem. Existence is first established in the presence of boundary conditions imposed only at the endpoint 0 employing an induction on the number of knots. The full theorem is proved by using the same continuity method coupled with induction on the form of the boundary conditions at 1.

We examine first the cases $n = 1$ and $n = 2$. Recalling the rank condition we find that the only cases not encompassed by the fundamental theorem without boundary conditions are the possibilities of $n = 2$, $k = l = 1$ and $n = 2$, $k = 1$, $l = 0$.

Consider $n = 2$ with $k = 1$, $l = 0$.

$$A_{10}M(0) + A_{11}M'(0) = 0$$

$$t_1 < \cdots < t_{2r+1}.$$

Proposition 2.7 demands

$$t_1 < \xi_1 < t_2.$$

Case 1. $A_{10} \neq 0$ entails $M(0) = \alpha M'(0)$ where $\alpha \geqq 0$. Thus

$$M(x) = x^2 - t_1^2 \frac{(\alpha + x)}{(\alpha + t_1)}, \qquad x \leqq t_1.$$

The determination thereafter is straightforward.

Case 2. $A_{10} = 0$ then $M'(0) = 0$. Thus

$$M(x) = (x - t_1)(x + t_1) \qquad x \leqq t_1.$$

We construct a Monospline $M(x)$ to vanish at the points $-t_1 < t_1 < t_2 < \cdots < t_{2r+1}$.

When $n = 2$, $k = l = 1$ we proceed as before using the fact that

$$t_1 < \xi_1 < t_2$$

$$t_{2r-2} < \xi_r < t_{2r}.$$

We may now stipulate throughout the remainder of the proof that $n \geqq 3$.

In this section we prove

THEOREM 5.1. *Let the $k \times n$ matrix $\| A_{\mu\nu}(-1)^\nu \|$ be SC_k and of rank k. Let the points $0 < t_1 < \cdots < t_{n-k+2r}$ be prescribed. Then there exists a unique monospline*

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1}$$

satisfying

$$\sum_{\nu=0}^{n-1} A_{\mu\nu} M^{(\nu)}(0) = 0, \qquad \mu = 1, \cdots, k$$

$$M(t_i) = 0 \qquad i = 1, \cdots, n - k + 2r.$$

PROOF. The validation of the theorem in the case $r = 0$ (the formulation reduces then to a linear problem), requires the inversion of a suitable matrix. The procedure is straightforward. We continue by induction assuming the validity of the theorem for the case of $r-1$ knots ($r \geqq 1$). Let $0 < t_1 < \cdots < t_{n-k+2r}$ be given. By the induction hypothesis there exists a monospline $M(x)$ with $r-1$ knots satisfying

$$\sum_{v=0}^{n-1} A_{\mu v} M^{(v)}(0) = 0, \qquad \mu = 1, 2, \cdots, k$$

$$M(t_i) = 0, \qquad i = 1, \cdots, n-k+2r-2.$$

We represent $M(x)$ in the form

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=}^{r-1} c_i (x - \xi_i)_+^{n-1}.$$

If

$$\xi \in [t_{n-k+2r-2}, t_{n-k+2r-1})$$

then

$$M(x;\xi) = M(x) - \frac{M(t_{n-k+2r-2})(x-\xi)_+^{n-1}}{(t_{n-k+2r-1} - \xi)^{n-1}}$$

is a monospline vanishing at the points $\{t_1, \cdots, t_{n-k+2r-1}\}$. Moreover, since $M(t_{n-k+2r-2}) > 0$, $M(t_{n-k+2r}; \xi)$ approaches $-\infty$ as ξ increases to $t_{n-k+2r-1}$. It follows that there exist $\bar{\xi} \in [t_{n-k+2r-2}, t_{n-k+2r-1})$ such that for $\bar{M}(x) = M(x; \bar{\xi})$ the following relations hold:

$$\sum_{j=0}^{n-1} A_{ij} \bar{M}^{(j)}(0) = 0, \qquad i = 1, 2, \cdots, k \tag{5.1}$$

$$\bar{M}(t_i) = 0, \qquad i = 1, \cdots, n-k+2r-1 \tag{5.2}$$

$$\bar{M}(t_{n-k+2r}) < 0. \tag{5.3}$$

We know from Proposition 2.1 that $\bar{M}$ vanishes at most $n-k+2r$ times on $(0, \infty)$. In light of (5.3), $\bar{M}$ actually admits a maximum number of zeros since $\bar{M}(t) \to +\infty$ when $t \to \infty$. Therefore there exists a unique $\bar{t} > t_{n-k+2r}$ for which

$$\bar{M}(\bar{t}) = 0.$$

On the basis of these facts and with the aid of Propositions 2.7 and 2.2, and Lemma 1.1, we infer that

$$(5.4)\qquad K^*\begin{pmatrix} t_1,\cdots,t_{n-k},\cdots,t_{n-k+2r-1} \\ i'_1,\cdots,i'_{n-k},\ \xi_1,\xi_1,\cdots,\xi_{r-1},\xi_{r-1},\bar{\xi} \end{pmatrix} > 0$$

for all indices $0 \leqq i_1 < \cdots < i_k \leqq n-1$ and also the relations

$$(5.5)\qquad c_i < 0, \qquad i = 1,\cdots,r-1$$

prevail.

We continue the proof in the form of some lemmas intermittent with discussion.

LEMMA 5.1. *There exists an* $\varepsilon > 0$ *and a unique family of monosplines*

$$(5.6)\qquad M(x,\tau) = x^n + \sum_{i=0}^{n-1} \lambda_i(\tau)x + \sum_{i=1}^{r} c_i(\tau)(x - \xi_i(\tau))_+^{n-1}$$

$0 < \xi_1(\tau) < \cdots < \xi_{r-1}(\tau) < \xi_r(\tau) = \tau < 1$ *fulfilling conditions* (5.1), (5.2), *and* (5.3) *for each* $\xi \in (\bar{\xi} - \varepsilon, \bar{\xi}]$.

PROOF. The map Φ of $E^{n-k+2r-1}$ into itself (cf. Section 2) defined by

$$(\lambda_0,\cdots,\lambda_{n-1},c_1,\xi_1,c_2,\xi_2,\cdots,c_{r-1},\xi_{r-1},c_r,\tau)$$

$$\to \left(\sum_{j=0}^{n-1} A_{1j}M^{(j)}(0),\cdots,\sum_{j=0}^{n-1} A_{kj}M^{(j)}(0), M(t_1),\cdots,M(t_{n-k+2r-1})\right)$$

has as its Jacobian evaluated with respect to all its variables except τ:

$$J(\Phi)\Big/\left(\prod_{i=1}^{r} c_i\right) =$$

$$\begin{vmatrix} A_{1\,0}0!,\cdots,A_{1,n-1}(n-1)! & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ A_{k,0}0!,\cdots,A_{k,n-1}(n-1)! & 0 & 0 & 0 \\ t_1^0,\cdots,\quad t_1^{n-1} & (t_1-\xi_1)_+^{n-1}, & \frac{\partial}{\partial\xi}(t_1-\xi_1)_+^{n-1},\cdots, & (t_1-\tau)_+^{n-1} \\ \vdots & \vdots & & \vdots \\ t_{n-k+2r-1}^0,\cdots,t_{n-k+2r-1}^{n-1}, & (t_{n-k+2r-1}-\xi_1)_+^{n-1}, & \cdots, & (t_{n-k+2r-1}-\tau)_+^{n-1} \end{vmatrix}$$

Expanding $J(\Phi)$ (compare with (2.7)) along the first k rows yields

$$J(\Phi) = (-1)^{k(k-1)/2} \left(\prod_{i=1}^{r} c_i\right) \sum_{0 \leqq i_1 < \cdots < i_k \leqq n-1} \left(\prod_{j=1}^{k} i_j!\right)$$

$$\times \ \tilde{A}\begin{pmatrix} 1, \cdots, k \\ i_1, \cdots, i_k \end{pmatrix} K^* \begin{pmatrix} t_1, \cdots, & & t_{n-k+2r-1} \\ i_1', \cdots, i_{n-k}', \xi_1, \xi_1, \cdots, \xi_{r-1}, \xi_{r-1}, \tau \end{pmatrix}$$

where $\| \tilde{A}_{\mu\nu} \| = \| A_{\mu\nu}(-1)^{\nu} \|$. Referring to (5.4) and (5.5) we see that $J(\Phi) \neq 0$ when $\tau = \bar{\xi}$. Appealing to the implicit function theorem affirms the conclusion of the lemma. The implicit function theorem further tells us that $\lambda_i(\tau)$, $c_i(\tau)$ and $\xi_i(\tau)$ are all continuously differentiable functions of the parameter τ.

Let ξ_0 be the infimum of all ξ where Lemma 5.1 is applicable on $(\xi_0, \bar{\xi}]$. We will establish that

$$\lim_{\tau \downarrow \xi_0 +} M(x, \tau) = M_0(x) \tag{5.7}$$

is the monospline which fulfills the requirements of Theorem 5.1. To this end, suppose first that the limit (5.7) exists and all the knots of $M_0(x)$ are in $(0, \infty)$. Then

$$\sum_{j=0}^{n-1} A_{ij} M_0^{(j)}(0) = 0, \qquad i = 1, \cdots, k$$

$$M_0(t_i) = 0 \qquad i = 1, \cdots, n-k+2r-1$$

$$M_0(t_{n-k+2r}) \leqq 0.$$

$M_0(t_{n-k+2r}) < 0$ then we can invoke Lemma 5.1 to extend the family of monosplines below ξ_0 contradicting the choice of ξ_0. Thus $M_0(t_{n-k+2r}) = 0$.

Before tackling the proof of (5.7) we need two further Lemmas.

Lemma 5.2. *Let $M(x, \tau)$ be the family of monosplines* (5.6) *satisfying*

$$(5.8)\qquad \begin{cases} \sum_{j=0}^{n-1} A_{ij}M^{(j)}(0,\tau) = 0, & i = 1,\cdots,k \\ M(t_i,\tau) = 0, & i = 1,\cdots,n-k+2r-1 \end{cases}$$

$$(5.9)\qquad M(t_{n-k+2r},\tau) < 0$$

for $\tau \in (\xi_0, \xi]$. *Then*

$$(5.10a)\qquad \frac{d}{d\tau} M(t_{n-k+2r},\tau) < 0$$

and

$$(5.10b)\qquad \frac{d\xi_i(\tau)}{d\tau} > 0, \qquad i = 1,2,\cdots,r-1$$

PROOF. Differentiating the relations (5.8) in τ (with $\xi(\tau) = \tau$) gives

$$0 = \sum_{j=0}^{n-1} A_{ij}\frac{d}{d\tau}M^{(j)}(0,\tau), \qquad i = 1,2,\cdots,k$$

$$(5.11)\qquad \begin{aligned} 0 = \frac{d}{d\tau}M(t_i,\tau) &= \sum_{\mu=0}^{n-1} (\mu!)\lambda'_\mu(\tau)\frac{t_i^\mu}{\mu!} + \sum_{\nu=1}^{r} c'_\nu(\tau)(t_i-\xi_\nu)_+^{n-1} \\ &+ \sum_{\nu=1}^{r} c_\nu(\tau)\frac{d\xi_\nu}{d\tau}\frac{\partial}{\partial\xi}(t_i-\xi_\nu)_+^{n-1}, \qquad i = 1,2,\cdots,n+2r-k-1 \end{aligned}$$

$$(5.12)\qquad \frac{d}{d\tau}M(t,\tau) = \sum_{\mu=0}^{n-1} (\mu!)\lambda'_\mu(\tau)\frac{\tau^\mu}{\mu!} + \cdots + \sum_{\nu=1}^{r} c_\nu(t)\frac{d\xi_\nu}{d\tau}\frac{\partial}{\partial\xi}(t_i-\xi_\nu)_+^{n-1}$$

with t a fixed value $> t_{n+2r-k-1}$. Regard the first $n+2r-1$ equations as a homogeneous system in the $n+2r$ variables

$$\lambda'_0(\tau),\cdots,\lambda'_{n-1}(\tau),$$

$$c'_1(\tau), c_1(\tau)\frac{d\xi_1}{d\tau},\cdots, c'_{r-1}(\tau), c_{r-1}(\tau)\frac{d\xi_{r-1}}{d\tau}, c'_r(\tau), c_r(\tau).$$

The matrix of the system is the $n + 2r - 1 \times n + 2r$ matrix

$$\begin{bmatrix} A_{10}0! & \cdots & A_{1,n-1}(n-1)! & 0 & 0 & 0 & 0 \\ \vdots & & & \vdots & \vdots & \vdots & \vdots \\ A_{k0}0! & \cdots & A_{k\,n-1}(n-1)! & 0 & 0 & 0 & 0 \\ t_1^0 & \cdots & t_1{}^{n-1} & (t_1-\xi_1)_+^{n-1} & \frac{\partial}{\partial\xi}(t_1-\xi_1)_+^{n-1}, & \cdots\ (t_1-\xi_r)_+^{n-1}, & \frac{\partial}{\partial\xi}(t_1-\xi_r)_+^{n-1} \\ \vdots & & & & \vdots & & \vdots \\ t_{n+2r-k-1}^0 & \cdots & t_{n+2r-k-1}^{n-1} & (t_{n+2r-k-1}-\xi_1)_+^{n-1} & \cdots & & \end{bmatrix}$$

The rank of this matrix is $(n+2r-1)$, (cf the analysis of (2.7)). Apart from a multiplicative constant the solution in (5.4) is proportional to the minors of the matrix obtained by eliminating successive columns and attributing to them the proper sign. More exactly, we obtain via the usual expansion (see 2.7)

$$\frac{(-1)^{2r-2\nu}c_\nu(\tau)\dfrac{d\xi_\nu}{d\tau}}{c_r(\tau)} = \tag{5.13}$$

$$\frac{\sum_{i_1<i_2<\cdots<i_r}(\prod i_\nu!)\tilde{A}\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix}K\begin{pmatrix}t_1,\cdots, & & , t_{2n+2r-k-1}\\ i'_1,\cdots,i'_{n-k},\xi_1,\xi_1,\cdots,\xi_{\nu-1},\xi_\nu,\xi_{\nu+1},\xi_{\nu+1},\cdots,\xi_r,\xi_r\end{pmatrix}}{\sum_{i_1<i_2<\cdots<i_r}(\prod i_\nu!)\tilde{A}\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix}K\begin{pmatrix}t_1,\cdots & & t_{n+2r-k-1}\\ i'_1,\cdots,i'_{n-k},\xi_1,\xi_1,\cdots,\xi_{r-1},\xi_{r-1},\xi_r\end{pmatrix}}$$

As $c_\nu(\tau)c_r(\tau)>0$ (both are negative) we deduce on the basis of (5.13) that

$$\frac{d\xi_\nu}{d\tau}>0, \qquad \nu=1,2,\cdots,r$$

and (5.10b) is proved.

Next append equation (5.12) to (5.11) and regard the known side as the vector $(0,0,\cdots,0,(d/d\tau)M(t,\tau))$. Solving for the variable $c_r(\tau)\,d\xi_r/d\tau = c_r(\tau)$ gives

$$c_r(\tau)$$

$$= \frac{dM(t,\tau)}{d\tau}\cdot\frac{(-1)^{k(k-1)/2}\sum\tilde{A}\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix}K\begin{pmatrix}t_1,\cdots, & & t_{n-k+2r-1}\\ i'_1,\cdots,i'_{n-k},\xi_1,\xi_1,\cdots,\xi_{r-1},\xi_{r-1},\tau\end{pmatrix}}{(-1)^{k(k-1)/2}\sum\tilde{A}\begin{pmatrix}1,\cdots,k\\ i_1,\cdots,i_k\end{pmatrix}K\begin{pmatrix}t_1,\cdots, & & t_{n-k+2r-1},t\\ i'_1,\cdots,i'_{n-k},\xi_1,\xi_1,\cdots,\xi_r,\xi_r\end{pmatrix}}$$

We know from Proposition 2.4 that necessarily $c_r(\tau)<0$. Therefore $dM(t,\tau)/d\tau < 0$ as claimed in (5.10a). In view of the fact that (5.3) holds for $M(t,\tau)$ there exists a unique $t(\tau)>t_{n-k+2r}$ such that $M(t(\tau),\tau)=0$.

COROLLARY 5.2. *Let $M(x;\tau)$ be as defined in Lemma 5.2. As τ decreases $\xi_1(\tau),\xi_2(\tau),\cdots,\xi_{r-1}(\tau)$ strictly decrease and the zero $t(\tau)$ strictly decreases.*

The final assertion is a direct consequence of (5.10a).

We return now to the proof of (5.7). We claim that it is enough to prove that there exists a constant $C>0$ such that

$$\begin{aligned} &|\lambda_i(\tau)|\leqq C, && i=0,1,\cdots,n-1\\ &c_i(\tau)\leqq C, && i=1,\cdots,r \qquad \tau\in(\xi_0,\xi]. \end{aligned} \tag{5.15}$$

For in this case there exists $\tau_\mu \to \xi_0$ such that

$$\lim_{\mu\to\infty} M(x,\tau_\mu) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x-\tilde{\xi}_i)_+^{n-1}$$

where

$$0 \leqq \tilde{\xi}_1 \leqq \cdots \leqq \tilde{\xi}_r$$

$$\lim_{\mu\to\infty} c_i'(\tau_\mu) = c_i$$

$$\lim_{\mu\to\infty} \lambda_i(\tau_\mu) = \lambda_i$$

$$\lim_{\mu\to\infty} \xi_i(\tau_\mu) = \tilde{\xi}_i$$

It follows that

$$M_0(t_i) = 0, \qquad i = 1,\cdots,n-k+2r-1$$

and

$$\lim_{\mu\to\infty} M^{(i)}(0,\xi_\mu) = M_0^{(i)}(0), \qquad i = 0,1,\cdots,n-2.$$

We wish to show that all the knots of M_0 are distinct and are located interior to $(0,\infty)$. Consider two cases. First, if $A_{i,n-1} = 0$, $i = 0,1,\cdots,n-1$ then M_0 also satisfies

$$\sum_{j=0}^{n-1} A_{ij} M_0^{(j)}(0) = 0, \qquad i = 1,\cdots,k.$$

It follows from Corollary 2.1 that M_0 has at least r knots in $(0,\infty)$. Thus

$$0 < \tilde{\xi}_1 < \cdots < \tilde{\xi}_r.$$

Secondly, if $A_{\mu,n-1} \neq 0$ then M_0 satisfies

$$\sum_{j=0}^{n-1}\left[A_{ij} - \frac{A_{\mu j}}{A_{\mu\, n-1}} A_{i,n-1}\right] M_0^{(j)}(0) = 0, \qquad \begin{matrix} i = 1,\cdots,k \\ i \neq \mu \end{matrix}$$

Since the matrix $\| A_{ij} - A_{\mu j}A_{i\,n-1}/A_{\mu,n-1} \|$ is SC_{k-1} and of rank $k-1$ we again infer that M_0 has at least r knots in $(0,\infty)$.

It remains to prove that the coefficients of $M(x,\tau)$ are uniformly bounded. Since $M(t_{n-k+2r},\tau) < 0$ there exists a $t(\tau) > t_{n-k+2r}$ noted earlier, satisfying

$$M(t(\tau),\tau) = 0.$$

According to Proposition 3.2 the coefficients of $M(x,\tau)$ are uniformly bounded on $(\xi_0,\bar{\xi}]$ provided that $t(\tau)$ is uniformly bounded on $(\xi_0,\bar{\xi}]$. But this is the case as assured by Corollary 5.2.

6. Fundamental theorem for two-sided boundary conditions

This section is devoted to completing the proof of Theorem 0.1 in the case of general boundary conditions at both end points. The method of analysis proceeds by induction on the form of the boundary conditions at the point 1. We treat first the case of 1 boundary condition.

Given $0 < t_1 < \cdots < t_{n-k+2r-1} < 1$ it is desired to show the existence of a monospline satisfying

$$\sum_{v=0}^{n-1} A_{\mu v} M^{(v)}(0) = 0, \qquad \mu = 1,\cdots,k$$

$$M(t_i) = 0, \qquad i = 1,\cdots,n-k+2r-1$$

$$\sum_{i=0}^{n-1} b_i M^{(i)}(1) = 0, \qquad \mu = 1,\cdots,l$$

where the b_i's are of a single sign and at least one is nonzero. Without loss of generality, we may assume

$$b_i \geqq 0, \qquad i = 0,1,\cdots,n-1.$$

Choose a point in $(t_{n-k+2r-1},1)$ and label it t_{n-k+2r}. During the course of the analysis of Section 5 we contructed a family of monosplines

$$M(x,\tau) = x^n + \sum_{i=1}^{n-1} \lambda_i(\tau)x^i + \sum_{i=1}^{r} c_i(\tau)(x-\xi_i(\tau))_+^{n-1}$$

where $0 < \xi_1(\tau) < \cdots < \xi_r(\tau) = \tau$, $\{\lambda_i(\tau)\}$, $\{c_i(\tau)\}$ are continuous and

$$\sum_{v=0}^{n-1} A_{\mu v} M^{(v)}(0,\tau) = 0, \qquad n = 1,\cdots,k \tag{6.1}$$

$$M(t_i,\tau) = 0, \qquad i = 1,\cdots,n-k+2r-1 \tag{6.2}$$

for τ in $[\xi_0, t_{n-k+2r-1})$. The point ξ_0 is uniquely determined by the additional requirement that

$$M(t_{n-k+2r},\xi_0) = 0. \tag{6.3}$$

$$M^{(i)}(1,\xi_0) > 0, \qquad i = 0,1,\cdots,n-1.$$

Moreover, recall that

$$\lim_{\tau \to t_{n-k+2r-1}} M^{(i)}(1,\tau) = -\infty .$$

Hence $\sum_{i=0}^{n-1} b_i M^{(i)}(1,\tau)$ changes sign on $[\xi_0, t_{n-k+2r-1})$. Thus there exists $\tau_0, \xi_0 < \tau_0 < t_{n-k+2r-1}$ satisfying $\sum_{i=0}^{n-1} b_i M^{(i)}(1,\tau_0) = 0$. The desired monospline is manifestly $M(x,\tau_0)$. Theorem 0.1 is established in the case $l = 1$. We discuss next the case of l boundary conditions, $l > 1$. Consider first the special situation where the boundary conditions at 1 have an associated coefficient matrix of the form

$$\begin{bmatrix} B_{10} & \cdots & B_{1,l-1} & 0 & \cdots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ B_{l0} & \cdots & B_{l,l-1} & 0 & \cdots & 0 \end{bmatrix}$$

The accompanying rank stipulation implies that the corresponding boundary conditions are equivalent to the conditions that M displays a zero of order l at 1. The desired result in this circumstance is an immediate consequence of Theorem 5.1. Now assume inductively that Theorem 0.1 is proved for l boundary conditions fulfilling Postulate II but where also $B_{\mu\nu} = 0$ $\nu \geqq m, \mu = 1, \cdots, l$ and $l \leqq m < n-1$. We extend the validity of Theorem 0.1 to the case of boundary conditions obeying Postulate II subject to the reduced restriction of

$$B_{\mu\nu} = 0, \ \nu \geqq m+1.$$

The matrix of the coefficients of the boundary forms is displayed as

$$B = \begin{bmatrix} B_{10} & \cdots & B_{1,m+1} & 0 & \cdots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ B_{l0} & \cdots & B_{l,m+1} & 0 & \cdots & 0 \end{bmatrix}$$

Assume first that the matrix

$$\tilde{B} = \begin{bmatrix} B_{10} & \cdots & B_{1,m+1} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ B_{l0} & \cdots & B_{l,m+1} \end{bmatrix}$$

is SSC_l; i.e., every $l \times l$ subdeterminant is strictly of one sign. This restriction will be removed later. For convenience we take the sign as $+1$.

Manifestly, we can rewrite the boundary conditions in terms of the equivalent matrix

$$B' = \begin{bmatrix} B'_{10} & \cdots & B'_{1m} & 0 & \cdots & 0 \\ & & & \cdot & & \cdot \\ & & & \cdot & & \cdot \\ & & & \cdot & & \vdots \\ B'_{l-1,0} & \cdots & B'_{l-1,m} & 0 & & \cdot \\ B'_{l0} & \cdots & B'_{lm} & B'_{lm,+1} & 0 \cdots & 0 \end{bmatrix} \tag{6.5}$$

where $B'_{l,m+1} > 0$ and

$$B'\begin{pmatrix} 1, \cdots, l \\ j_1, \cdots, j_l \end{pmatrix} = (\operatorname{sign} B_{l,m+1}) \cdot B \begin{pmatrix} 1, \cdots, l \\ j_1, \cdots, j_l \end{pmatrix} \tag{6.6}$$

for all $0 \leqq j_1 < \cdots < j_l \leqq n-1$. Therefore without loss of generality we may assume B has this form. Now observe that the matrices $\| (-1)^\nu A_{\mu\nu} \|$ and

$$\hat{B} = \begin{bmatrix} B_{10} & \cdots & B_{1m} & 0 \cdots 0 \\ \cdot & & \cdot & \cdot \quad \cdot \\ \cdot & & \cdot & \cdot \quad \cdot \\ \cdot & & \cdot & \cdot \quad \cdot \\ B_{l0} & \cdots & B_{lm} & 0 \cdots 0 \end{bmatrix}$$

fulfill the requirements of Postulate II. Indeed if $2r < l$ there exists indices $0 \leqq i_1 < \cdots < i_k \leqq n-1$, $0 \leqq j_1 < \cdots < j_l \leqq n-1$

$$A\begin{pmatrix} 1, \cdots, k \\ i_1, \cdots, i_k \end{pmatrix} \neq 0, \qquad B\begin{pmatrix} 1, \cdots, l \\ j_1, \cdots, j_l \end{pmatrix} \neq 0 \tag{6.7}$$

with $j_\mu \leqq i'_{n+2r-k-l+\mu}$, $\mu = 1, \cdots, l-2r$. But certainly

$$j_\mu^0 \leqq j_\mu \quad \text{for } j_\mu^0 = \mu - 1 \tag{6.8}$$

and

$$\tilde{B}\begin{pmatrix} 1, & \cdots, l \\ j_1^0, & \cdots, j_l^0 \end{pmatrix} = B\begin{pmatrix} 1, & \cdots, l \\ j_1^0, & \cdots, j_l^0 \end{pmatrix} \neq 0$$

since $\tilde{B}$ is SSC_l.

Therefore by the induction hypothesis we are assured of the existence of a monospline

$$M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i(x - \xi_i)_+^{n-1}$$

satisfying

$$0 < \xi_1 < \cdots < \xi_r < 1$$

$$\sum_{\nu=0}^{n-1} A_{\mu\nu} M^{(\nu)}(0) = 0, \qquad \mu = 1,2,\cdots,k$$

$$M(t_i) = 0, \qquad i = 1,\cdots,n+2r-k-l$$

$$\sum_{\nu=0}^{m} B_{\mu\nu} M^{(\nu)}(1) = 0, \qquad \mu = 1,\cdots,l$$

Note that $M(x)$ satisfies all the constraints necessary to advance the induction to the matrix B except for the last boundary condition at 1 which specifically requires that the quantity

$$\sum_{\nu=0}^{n-1} B_{l\nu} M^{(\nu)}(1) = \sum_{\nu=0}^{m} B_{l\nu} M^{(\nu)}(1) + B_{l,m+1} M^{(m+1)}(1) = B_{l,m+1} M^{(m+1)}(1)$$

should vanish.

The monospline $M(x)$ will serve as a starting point for the continuity method used to construct the desired monospline. It is convenient to divide the subsequent analysis into a series of lemmas.

Lemma 6.1. *There exists an* $\varepsilon > 0$ *and a family of monosplines*

$$M(x,\tau) = x^n + \sum_{i=0}^{n-1} \lambda_i(\tau) x^i + \sum_{i=1}^{r} c_i(\tau)(x - \xi_i(\tau))_+^{n-1}$$

determined for each τ *in* $[\xi_r, \xi_r + \varepsilon)$ *with the properties*

$$0 < \xi_1(\tau) < \cdots < \xi_r(\tau) = \tau < 1$$

(6.9) (i) $M(x,\xi_r) = M(x)$

(6.10) (ii) $\sum_{\nu=0}^{n-1} A_{\mu\nu} M^{(\nu)}(0,\tau) = 0, \qquad \mu = 1,\cdots,k$

(6.11) (iii) $M(t_i,\tau) = 0, \qquad i = 1,\cdots,n+2r-k-l$

$$\text{(6.12)} \qquad \text{(iv)} \sum_{\nu=0}^{n-1} B_{\mu\nu} M^{(\nu)}(1,\tau) = 0, \qquad \mu = 1,\cdots,l-1$$

$$\text{(6.13)} \qquad \text{(v)} \sum_{\nu=0}^{n-1} B_{l\nu} M^{(\nu)}(1,\tau) > 0,$$

$$\text{(6.14)} \qquad \text{(vi)}\ S^{+}(M(1,\tau), M'(1,\tau),\cdots, M^{(n)}(1,\tau)) = l.$$

PROOF. Consider the map $\Psi: E^{n+2r} \to E^{n+2r-1}$

$$\Psi: (\lambda_0,\cdots,\lambda_{n-1}, c_1, \xi_1,\cdots, c_r, \xi_r)$$

$$\to \Bigg[\underbrace{\sum_{\nu=0}^{n-1} A_{\mu\nu} M^{(\nu)}(0),}_{k} \quad \underbrace{M(t_i),}_{n+2r-k-l} \quad \underbrace{\sum_{\nu=0}^{n-1} B_{\mu\nu} M^{(\nu)}(1)}_{l-1}\Bigg].$$

Then

$$\Psi(\lambda_0,\cdots,\lambda_{n-1}, c_1, \xi_r,\cdots, c_r, \xi_r) = 0.$$

The Jacobian of Ψ with respect to all the variables except ξ_r (cf. formula (2.7)) is

$$J \equiv (-1)^{k(k-1)/2} \left(\prod_{i=1}^{r} c_i\right) \sum_{\substack{0 \leqq i_1 < \cdots < i_k \leqq n-1 \\ 0 \leqq j_1 < \cdots < j_{l-1} \leqq m}} \tilde{A}\begin{pmatrix} 1,\cdots,k \\ i_1,\cdots,i_k \end{pmatrix} B\begin{pmatrix} 1,\cdots,l-1 \\ j_1,\cdots,j_{l-1} \end{pmatrix}$$

$$\times\ K\begin{pmatrix} t_1,\cdots,t_N, j_1,\cdots,j_{l-1} \\ i'_1,\cdots,i'_{n-k}, \xi_1,\xi_1,\cdots,\xi_{r-1},\xi_{r-1},\xi_r \end{pmatrix}$$

$$(N = n + 2r - k - l,\ \| \tilde{A}_{\mu\nu} \| = \| A_{\mu\nu}(-1)^{\nu} \|).$$

Since B is SSC_l by assumption we invoke Proposition 2.7, Corollary 2.2 and (6.8) to conclude that $J \neq 0$ at $(\lambda_0,\cdots,\lambda_{n-1}, c_1, \xi_1,\cdots, c_r, \xi_r)$. Now an appeal to the implicit function theorem establishes (6.9)–(6.12).

We next examine the expression

$$\sum_{\nu=0}^{n-1} B_{l\nu} M^{(\nu)}(1) = \sum_{\nu=0}^{m} B_{l\nu} M^{(\nu)}(1) + B_{l,m+1} M^{(m+1)}(1).$$

Since $B_{l\nu} = 0$ for $\nu > m + 1$ we refer to Lemma 2.1 to infer that $M^{(m+1)}(1) > 0$ and the validity of (6.13) is deduced by direct continuity considerations. Our final task is to prove (6.14). From Proposition 2.1 we know that

$$S^{+}(M(1,\tau),\cdots, M^{(n)}(1,\tau)) \leqq l.$$

On the other hand, Lemma 2.1 informs us that

$$l - 1 \leqq S^{+}(M(1,\tau),\cdots, M^{(n)}(1,\tau)).$$

To complete the analysis it suffices to verify that $(-1)^l M(1) > 0$. To this end, note that $(-1)^l M(1) \geqq 0$ because $S^+(M(1),\cdots,M^{(n)}(1)) = l$ and $M^{(n)}(1) > 0$ prevail. Moreover, $M(1)$ cannot vanish for otherwise M satisfies $l+1$ boundary conditions at the point 1 with associated coefficient matrix

$$\begin{bmatrix} 1 & 0 \cdots & 0 & 0 & \cdots 0 \\ B_{10} & \cdots & B_{1m} & 0 & \cdots 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ B_{l0} & \cdots & B_{lm} & B_{lm+1} & \cdots 0 \end{bmatrix}$$

which is clearly SC_{l+1} of rank $l+1$. This information is incompatible with the conclusion of Proposition 2.2 and the fact that $Z(M) = n + 2r - k - l$. The proof of Lemma 6.1 is complete.

Define α_0 as the supremum taken over all $\alpha < 1$ where $M(x,\tau)$ possesses the properties listed in Lemma 6.1 for all τ in $[\xi_r,\alpha)$. We will show that the limit as $\tau \to \alpha_0$ of $M(x,\tau)$ exists and this limit function determines the desired monospline fulfilling the requirements of Theorem 0.1.

The next lemma formalizes the outcome of this limit process.

LEMMA 6.2. *We have $\alpha_0 < 1$ (α_0 is defined immediately above) and*

$$\lim_{\tau \to \alpha_0} M(x,\tau) \equiv M(x) = x^n + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}$$

$$0 < \xi_1 < \cdots < \xi_r = \alpha_0 < 1.$$

Moreover, $M(x)$ satisfies

$$\sum_{\nu=0}^{n-1} A_{\mu\nu} M^{(\nu)}(0) = 0, \qquad \mu = 1,\cdots,k$$

$$M(t_i) = 0, \qquad i = 1,\cdots,n+2r-k-l$$

$$\sum_{\nu=0}^{n-1} B_{\mu\nu} M^{(\nu)}(1) = 0, \qquad \mu = 1,\cdots,l$$

PROOF. Our first need is to establish that the coefficients of the family of monosplines

$$M(x,\tau) = x^n + \sum_{i=0}^{n-1} \lambda_i(\tau) x^i + \sum_{i=1}^{r} c_i(\tau)(x - \xi_i(\tau))_+^{n-1}$$

are uniformly bounded for τ in $[\xi_r, \alpha_0)$. This assertion ensues on the basis of Proposition 3.2 in the case $2r \geqq l$. When $2r < l$ boundedness is proved employing an argument paralleling that of Proposition 3.3. In fact, we know from Proposition 3.1 that $\{c_i(\tau)\}_{i=1}^r$ are bounded on $[\xi_r, \alpha_0)$. We now show that $\{\lambda_i(\tau)\}_{i=0}^{n-1}$ are also bounded. Set

$$Y(x,\tau) = x^n + \sum_{i=1}^{r} c_i(\tau)(x - \xi_i(\tau))_+^{n-1}$$

$$P(x,\tau) = \sum_{i=0}^{n+2r-2} \lambda_i(\tau)x^i, \qquad (\lambda_i(\tau) \equiv 0, i \geqq n).$$

Then

$$M(x,\tau) = Y(x,\tau) + P(x,\tau)$$

and there exists a constant C such that

$$|H_\mu(\tau)| \equiv \left|\sum_{v=0}^{n-1} A_{\mu v} Y^{(v)}(0,\tau)\right| \leqq C, \qquad \mu = 1,\cdots,k$$

$$|L_\mu(\tau)| \equiv \left|\sum_{v=0}^{n-1} B_{\mu v} Y^{(v)}(0,\tau)\right| \leqq C, \qquad \mu = 1,\cdots,l$$

$$\max_{0 \leqq x \leqq 1} |Y(x,\tau)| \leqq C$$

for $\tau \in [\xi_r, \alpha_0)$.

In view of (6.10)–(6.12) we have

$$\sum_{v=0}^{n-1} A_{\mu v} P^{(v)}(0,\tau) = -H_\mu(\tau), \qquad \mu = 1,\cdots,k$$

$$P(t_i,\tau) = -Y(t_i,\tau), \quad i = 1,\cdots,n+2r-k-l$$

$$\sum_{v=0}^{n-1} B_{\mu v} P^{(v)}(1,\tau) = -L_\mu(\tau), \qquad \mu = 1,\cdots,l-1.$$

These are $n+2r-1$ linear equations in the $n+2r-1$ "unknowns" $\{\lambda_i(\tau)\}_{i=0}^{n+2r-2}$ with the inhomogeneous right hand side consisting of quantities uniformly bounded in τ. Thus, we need only check that the determinant Δ of the system (which is independent of τ) is nonzero. Expanding this determinant Δ (cf. (2.7) of Section 2) gives

$$|\Delta| \geqq \gamma K \begin{pmatrix} t_1, \cdots, t_{n+2r-k-l} & j_1^0, \cdots, j_{l-1}^0 \\ i_1', \cdots, & i_{n+2r-k-1}' \end{pmatrix}$$

where $0 \leqq i_1 < \cdots < i_k \leqq n-1$ and $j_\mu^0 = \mu - 1$ ($\mu = 1 \cdots l$) and $\gamma > 0$. But as previously pointed out (see 6.7)) we have

$$j_\mu^0 \leqq i'_{n+2r-k-l+\mu}, \qquad \mu = 1 \cdots l-2r$$

and so $\Delta \neq 0$ by virtue of Proposition 2.7 and Lemma 1.1.

The boundedness of the coefficients is proved. Since the coefficients are bounded there exists a sequence $\tau_\mu \to \alpha_0$ such that the limit relations

$$\begin{aligned} \lambda_i(\tau_\mu) &\to \lambda_i^*, & i &= 0,1,\cdots,n-1 \\ c_i(\tau_\mu) &\to c_i^*, & i &= 1,\cdots,r \\ \xi_i(\tau_\mu) &\to \xi_i^*, & i &= 1,\cdots,r \end{aligned}$$

persist.

Set

$$M_*(x) = x^n + \sum_{i=0}^{n-1} \lambda_i^* x^i + \sum_{i=1}^{r} c_i^*(x - \xi_i^*)_+^{n-1}$$

$$0 \leqq \xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_r = \alpha_0 \leqq 1.$$

Then clearly for all x

$$\lim_{\mu \to \infty} M^{(i)}(x,\tau_\mu) = M_*^{(i)}(x), \qquad i = 0,1,\cdots,n-2. \tag{6.15}$$

We will next perform a computation to show that the knots increase as τ increases. Direct differentiation yields

$$\begin{aligned} \frac{\partial}{\partial \tau} M(x,\tau) &= \sum_{i=0}^{n-1} \lambda_i'(\tau) x^i + \sum_{i=1}^{r} c_i'(\tau)(t - \xi_i)_+^{n-1} \\ &\quad + \sum_{i=1}^{r} c_i(\tau)\xi_i'(\tau) \frac{\partial}{\partial \xi_i}(t - \xi_i)_+^{n-1} \end{aligned} \tag{6.16}$$

and we also obtain

$$\begin{aligned} &\sum_{\nu=0}^{n-1} A_{\mu\nu} \frac{\partial}{\partial \tau} M^{(\nu)}(0,\tau) = 0, & \mu &= 1,\cdots,k \\ &\frac{\partial}{\partial \tau} M(t_i,\tau) = 0, & i &= 1,\cdots,n+2r-k-l \\ &\sum_{\nu=0}^{n-1} \bar{B}_{\mu\nu} \frac{\partial}{\partial \tau} M^{(\nu)}(1,\tau) = \delta_{\mu 0} \frac{\partial}{\partial \tau} M(1,\tau), & \mu &= 0,1,\cdots,l-1 \end{aligned} \tag{6.17}$$

where

$$\bar{B} = \begin{bmatrix} 1 & 0 \cdots 0 & 0 \cdots 0 \\ B_{10} & B_{im} & 0 \cdots 0 \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ B_{l-1,0} & B_{l-1,m} & 0 \cdots 0 \end{bmatrix}$$

Regard the equations (6.17) as a system of $n + 2r$ linear equations in the $n + 2r$ "unknowns" $\lambda_0'(\tau), \cdots \lambda_{n-1}'(\tau), c_1'(\tau), c_1(\tau)\xi_1'(\tau), \cdots, c_{r-1}'(\tau), c_{r-1}(\tau)\xi_{r-1}'(\tau), c_r(\tau)\xi_r'(\tau)$ (recall that $\xi_r(\tau) = \tau$). Solving gives

$$\begin{aligned} c_i(\tau)\xi_i'(\tau) &= (-1)^{n+2r-k-l+k+n+2i+1} \frac{\partial}{\partial \tau} M(1,\tau)\frac{\Delta_i}{\Delta} \\ &= (-1)^{l-1} \frac{\partial}{\partial\tau} M(1,\tau)\frac{\Delta_i}{\Delta}, \qquad i = 1,2,\cdots,r \end{aligned}$$

$$\Delta_i = (-1)^{k(k-1)/2} \sum_{\substack{0 \leqq i_1 < \cdots < i_k \leqq n-1 \\ 0 \leqq j_1 < \cdots < j_{l-1} \leqq m}} \left(\prod_{j=1}^{k} i_j!\right) \tilde{A}\begin{pmatrix} 1, \cdots, k \\ i_1, \cdots, i_k \end{pmatrix} B\begin{pmatrix} 1, \cdots, l-1 \\ j_1, \cdots, j_{l-1} \end{pmatrix}$$

$$(6.18) \qquad \times K \begin{pmatrix} t_1, \cdots, t_N, j_1, \cdots, j_{l-1} \\ i_1', \cdots, i_{n-k}', \xi_1, \xi_1, \cdots, \xi_{i-1}, \xi_{i-1}, \xi_i, \xi_{i+1}, \xi_{i+1} \cdots, \xi_r \end{pmatrix}$$

$$\Delta = (-1)^{k(k-1)/2} \sum_{\substack{0 \leqq i_1 < \cdots < i_k \leqq n-1 \\ 0 \leqq j_1 < \cdots < j_{l-1} \leqq m}} \left(\prod_{j=1}^{k} i_j!\right) \tilde{A}\begin{pmatrix} 1, \cdots, k \\ i_1, \cdots, i_k \end{pmatrix} B\begin{pmatrix} 1, \cdots, l-1 \\ j_1, \cdots, j_{l-1} \end{pmatrix}$$

$$\times K\begin{pmatrix} t_1, \cdots, t_N, 0, j_1, \cdots, j_{l-1} \\ i_1', \cdots, i_{n-k}' \ \xi_1 \ \xi_1, \cdots \ \xi_r, \xi_r \end{pmatrix}$$

With the help of Proposition 2.4 and Lemma 1.1 we find that $(-1)^{k(k-1)/2}\varepsilon_k\Delta > 0$ (when $2r < l$ use the indices $j_\mu = \mu$, $\mu = 1, \cdots, l-1$) and $(1)^{k(k-1)/2}\varepsilon_k\varepsilon_i\Delta_i \geqq 0$ by virtue of Lemma 1.1 where $\varepsilon_k = \pm 1$ is the sign of the kth order minor of $\tilde{A}$. Also we know according to Proposition 2.3, that $c_i(\tau) < 0$, $i = 1, \cdots, r$. These facts plus that $\xi_r(\tau)$ for $i = r$ in (6.18) lead to the inequality

$$(6.19) \qquad (-1)^l \frac{\partial}{\partial\tau} M(1,\tau) \geqq 0, \qquad \tau \in (\xi_r, \alpha_0)$$

and then in (6.18) with $i = 1,2,\cdots,r-1$ the result

$$(6.20) \qquad \xi_i'(\tau) \geqq 0, \qquad i = 1, \cdots, r, \qquad \tau \in (\xi_r, \alpha_0).$$

The relations (6.20) imply, in particular, that $0 < \xi_1 \leqq \xi_1^*$ and consequently

$$\sum_{i=0}^{n-1} A_{\mu\nu} M_*^{(\nu)}(0) = 0, \qquad \mu = 1, \cdots, k.$$

Owing to (6.15) and because of the stipulation $B_{\mu,n-1} = 0$, $\mu = 1, \cdots, l-1$ we deduce that $M_*(x)$ further satisfies the equations

$$\begin{aligned} &\sum_{\nu=0}^{n-1} B_{\mu\nu} M_*^{(\nu)}(1) = 0, \qquad && \mu = 1, \cdots, l-1 \\ &M_*(t_i) = 0, && \mu = 1, \cdots, n+2r-k-l. \end{aligned} \tag{6.21}$$

Now let r' denote the number of distinct knots of M inside $(0,1)$. Proposition 2.1 and Lemma 1.2 affirm that

$$n + 2r - k - l \leqq n + 2r' - k - l + 1$$

or

$$2r \leqq 2r' + 1.$$

Therefore $r' = r$ and so

$$0 < \xi_1 < \cdots < \xi_r = \alpha_0 < 1.$$

It remains to demonstrate that

$$\sum_{\nu=0}^{n-1} B_{i\nu} M_*^{(\nu)}(1) = 0.$$

Suppose to the contrary that $\sum_{\nu=0}^{n-1} B_{l\nu} M_*^{(\nu)}(1) > 0$. If we can show that $S^+(M_*(1), \cdots, M_*^{(n)}(1)) = l$ we then appeal to the implicit function theorem as in Lemma 6.1 to extend $\xi_r = \alpha_0$ to the right. This provides a contradiction to the the definition of α_0. However, Proposition 2.1 and Lemma 2.1 in conjunction with (6.20)–(6.22) imply

$$l - 1 \leqq S^+(M_*(1), \cdots, M_*^{(n)}(1)) \leqq l. \tag{6.22}$$

But (6.19) implies

$$(-1)^l M(1, \tau) \geqq (-1)^l M(1).$$

It follows that

$$(-1)^l M_*(1) \geqq (-1)^l M(1).$$

Recalling that $(-1)^l M(1) > 0$, a fact noted during the proof of Lemma 6.1, we obtain the identical inequality for $M_*(1)$. Since $M_*^{(n)}(1) > 0$, it follows from (6.22) that

$$S^{+}(M_{*}(1),\cdots,M_{*}^{(n)}(1)) = l.$$

The proof of Lemma 6.2 is complete.

The conclusion of Lemma 6.2 is synonomous with that of Theorem 0.1 except that we need to relax the SSC_l hypothesis on the boundary conditions. Suppose that $\| B_{\mu\nu} \|_{\substack{\mu=1,\cdots,l \\ \nu=0,\cdots,m+1}}$ is merely SC_l. By a standard approximation procedure there exists for each $\sigma > 0$, $B(\sigma) = \| B_{\mu\nu}(\sigma) \|_{\substack{\mu=1,\cdots,l \\ \nu=0,1,\cdots,m+1}}$ which is SSC_l and such that $\lim_{\tau\to\infty} B(\sigma) = B$ (see [7, p. 22]). The indices of Postulate II for $\| B_{\mu\nu} \|$ certainly apply for $B(\sigma)$. Hence there exists

$$M_{\sigma}(x) = x^n + \sum_{i=0}^{n-1} \lambda_i(\sigma)x^i + \sum_{i=1}^{r} c_i(\sigma)(x - \xi_i(\sigma))_+^{n-1}$$

fulfilling the conditions

$$\sum_{\nu=0}^{n-1} A_{\mu\nu} M_{\sigma}^{(\nu)}(0) = 0, \qquad \mu = 1,\cdots,k$$

$$M_{\sigma}(t_i) = 0, \qquad i = 1,\cdots,n+2r-k-l$$

$$\sum_{\nu=0}^{n-1} B_{\mu\nu} M_{\sigma}^{(\nu)}(1) = 0, \qquad \mu = 1,\cdots,l.$$

If the coefficients of M_σ are uniformly bounded then we can argue as in Lemma 6.2 and clearly $\lim_{\sigma\to\infty} M_\sigma(x)$ furnishes the desired monospline.

In the case that $2r \geqq l$ boundedness of $\{\lambda_i(\sigma)\}$, $\{c_i(\sigma)\}$ follows along the lines of Proposition 3.2. When $2r < l$ we reason as in Proposition 3.3 or Lemma 6.2. Indeed, let $P_\sigma(t) = \sum_{i=0}^{n+2r-1} \lambda_i(\sigma)t^i$, $\lambda_i(\sigma) \equiv 0$, $i \geqq n$. Then P_σ satisfies an inhomogeneous system of linear equations. The inhomogeneous terms are uniformly bounded in σ, and if Δ_σ denotes the determinant of the system then

$$\Delta_\sigma = (-1)^{(k(k-1))/2} \sum_{\substack{0 \leqq j_1 < \cdots < j_l \leqq m-1 \\ 0 \leqq i_1 < \cdots < i_k \leqq n-1}} \left[\prod_{j=1}^{l} i_j! \right] \tilde{A} \begin{bmatrix} 1,\cdots,k \\ i_1,\cdots,i_k \end{bmatrix} B_\sigma \begin{bmatrix} 1,\cdots,l \\ j_1,\cdots,j_l \end{bmatrix}$$

$$\times K \begin{pmatrix} t_1,\cdots,t_N \; j_1,\cdots,j_l \\ i'_1,\cdots,i'_{n+2r-k} \end{pmatrix} \qquad (N = n+2r-k-l) .$$

which is bounded away from zero. The remainder of the argument proceeds mutatis mutandis.

7. Applications and extensions

As an application of Theorem 0.1 we establish the existence of certain types of quadrature formulas exhibiting "*double precision*". These formulas can be interpreted as constituting extensions of the classical quadrature formulas of Gauss, Rado and Lobbato. The connection between "double precision" quadrature formulas was first indicated by Schoenberg [16], (see also [9]).

Let n, k, l, r be natural numbers satisfying $n \leqq k + l + 2r$, and special indices

$$0 \leqq i_1 < \cdots < i_k \leqq n-1$$

$$0 \leqq j_1 < \cdots < j_l \leqq n-1$$

obeying the relations

$$j_\mu \leqq i'_{\nu+2r} \quad \nu = 1, \cdots, n-l-2r \tag{7.1}$$

when $n > l + 2r$ and $s = k + l + 2r - n$. Consider arbitrary but fixed points

$$0 < x_1 < \cdots < x_s < 1.$$

THEOREM 7.1. *There exists a unique quadrature formula*

$$\int_0^1 f(x)\,dx \sim \sum_{\mu=1}^{k} A_\mu f^{(i_\mu)}(0) + \sum_{\mu=1}^{l} B_\mu f^{(j_\mu)}(1) + \sum_{i=1}^{r} \lambda_i f(\xi_i) \tag{7.2}$$

with $0<\xi_1< \cdots < \xi_r < 1$; i.e., A_μ, B_ν, λ and ξ_i *exist such that equality prevails in* (7.2) *for the family of functions* $\{1, \cdots, x^{n-1}, (x-x_1)_+^{n-1}, \cdots, (x-x_s)_+^{n-1}\}$.

Moreover, the weights $\{\lambda_i\}_{i=1}^r$ *are all positive and the signs of* $\{A_\mu\}$, $\{B_\mu\}$ *are computed by the formula* $S^+(\alpha) = n-k$, $S^+(\beta) = n-l$ *where*

$$\alpha_i = \begin{cases} 0, & \text{if } i = n-1-i'_\mu, \quad \mu = 1, \cdots, n-k \\ (-1)^{n-1}A_\mu, & \text{if } i = n-1-i_\mu, \quad \mu = 1, \cdots, k \\ 1, & \text{if } i = n \end{cases}$$

(7.3)

$$\beta_i = \begin{cases} 0, & \text{if } i = n-1-j'_\mu, \quad \mu = 1, \cdots, n-l \\ (-1)^{j_\mu}B_\mu, & \text{if } i = n-1-j_\mu, \quad \mu = 1, \cdots, l \\ 1, & \text{if } i = n. \end{cases}$$

PROOF. Consider a monospline

$$M(x) = \frac{x^n}{n!} + \sum_{i=0}^{n-1} b_\nu x^\nu + \sum_{i=1}^{r} c_i (x-\xi_i)_+^{n-1}.$$

Let f be of continuity class C^n, then integration by parts produces the identity

$$\int_0^1 f(x)\,dx = \sum_{j=0}^{n-1} (-1)^j M^{(n-1-j)}(1) f^{(j)}(1) - \sum_{j=0}^{n-1} (-1)^j M^{(n-j-1)}(0) f^{(j)}(0)$$

(7.4)

$$- \sum_{k=1}^{r} c_k (n-1)!\, f(\xi_k) + (-1)^n \int_0^1 M(x) f^{(n)}(x)\,dx.$$

Now determine $M(x)$ as the monospline of degree n involving r knots in $(0,1)$ and satisfying the conditions

(7.5) $$M^{(n-1-i'_\mu)}(0) = 0, \qquad \mu = 1,\cdots,n-k$$

(7.6) $$M(x_i) = 0, \qquad i = 1,\cdots,s$$

(7.7) $$M^{(n-1-j'_\mu)}(1) = 0, \qquad \mu = 1,\cdots,n-l\,.$$

The existence of such a monospline is assured by virtue of Theorem 0.1 and the stipulations of (7.1) and the fact of $n + 2r - (n-k) - (n-l) = s$. We write

$$\tilde{M}(x) = \frac{x^n}{n!} + \sum_{i=0}^{n-1} \lambda_i x^i + \sum_{i=1}^{r} c_i (x - \xi_i)_+^{n-1}.$$

Substituting in (7.4) yields apart from the integral on the right the quadrature approximation

$$\int_0^1 f(x)\,dx \sim \sum_{\mu=1}^{k} (-1)^{i_\mu} \tilde{M}^{(n-1-i_\mu)}(0) f^{(i_\mu)}(0)$$

(7.8) $$- \sum_{\mu=1}^{l} (-1)^{j} \tilde{M}^{(n-1-j_\mu)}(1) f^{(j_\mu)}(1)$$

$$- (n-1)! \sum_{i=1}^{r} c_i f(\xi_i) \equiv Q(f).$$

This quadrature formula is manifestly exact for the functions $\{1, x, \cdots, x^{n-1}\}$ by the nature of the remainder term in (7.4). Making the obvious identifications of $\{\lambda_\mu\}$, $\{A_\mu\}$ and $\{B_\mu\}$ we see that the relations of (7.3) are valid according to Proposition 2.6.

To show that the quadrature formula (7.8) becomes equality for the function $(x - x_i)_+^{n-1}$ is equivalent to showing equality for the function $(x_i - x)_+^{n-1}$ since

$$(x - x_i)_+^{n-1} + (-1)^n (x_i - x)_+^{n-1} = (x - x_i)^{n-1}\ .$$

With the help of (7.6) we obtain

$$\int_0^1 (x_i - x)_+^{n-1} dx$$

$$= \int_0^{x_i} (x_i - x)^{n-1} dx = \frac{-(x_i - x)^n}{n}\Big|_0^{x_i}$$

$$= \frac{x_i^n}{n}$$

$$= (n-1)! \left[-\sum_{\mu=1}^{k} \lambda_{n-1-i_\mu} x_i^{n-1-i_\mu} - \sum_{j=1} c_j (x_i - \xi_j)_+^{n-1} \right]$$

$$= (n-1)! \left[-\sum_{\mu=1}^{k} \lambda_{n-1-i_\mu} (-1)^{i_\mu} \frac{(n-1-i_\mu)!}{(n-1)!} \frac{d^{i_\mu}}{dx^{i_\mu}} (x_i - x)^{n-1} \Big|_{x=0} \right.$$

$$\left. - \sum_{j=1}^{r} c_j (x_i - \xi_j)_+^{n-1} \right]$$

$$= Q((x_i - x)_+^{n-1}).$$

Theorem 7.1 is also proven in [10] when $s = 0$. In [10] the nodes of the quadrature formulae are identified as the zeros of a certain extremal polynomial.

The results of this paper extend to the case of Extended Complete Tchebycheff systems. We merely state the result without entering into details.

If $0 < w_i \leqq w_i(x) \leqq \bar{w}_i < \infty$ for $x \in (-\infty, +\infty)$ we define

$$U_0(x) = w_0(x)$$

$$U_1(x) = w_0(x) \int_0^x w_1(\xi_1) d\xi_1$$

$$\vdots$$

$$U_n(x) = w_0(x) \int_0^x w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2) \cdots \int_0^{\xi_{n-1}} w_n(\xi_n) d\xi_n \cdots d\xi_1,$$

and

$$D_j \Phi = \frac{d}{dt} \left[\frac{\Phi(t)}{w_j(t)} \right], \qquad j = 0, 1, \cdots, n .$$

A Tchebycheffian monospline with respect to $\{U_i\}_{i=0}^n$ has the form

(7.9) $$U_n(x) + \sum_{i=0}^{n-1} \lambda_i U_i(x) + \sum_{i=1}^{r} c_i \Phi_{n-1}(x,\xi_i)$$

where

$$\Phi_n(x,\xi) = \begin{cases} w_0(x)\int_\xi^x w_1(\xi_1)\int_\xi^{\xi_1} w_2(\xi_2)\cdots\int_\xi^{\xi_{n-1}} w_n(\xi_n)\,d\xi_n\cdots d\xi_1, & \xi \leqq x \\ 0 & \xi > x \end{cases}$$

If $\| A_{\mu\nu} \|$ and $\| B_{\mu\nu} \|$ satisfy Postulate II and (7.9) satisfies boundary conditions of the form

$$\sum_{\nu=0}^{n-1} A_{\mu\nu}(D^\nu M)(0) = 0, \qquad \mu = 1,\cdots,k$$

$$\sum_{\nu=0}^{n-1} B_{\mu\nu}(D^\nu M)(1) = 0, \qquad \mu = 1,\cdots,l$$

where

$$D^\nu = D_\nu D_{\nu-1}\cdots D_1 D_0, \qquad D_0 = \text{Identity operator}$$

then Theorem 0.1 persists.

An important particular case is the system $\{1,\cdots,x^{n-1},f(x)\}$ where $f^{(n)}(x) > 0$. Hence the fundamental theorem applies if x^n is replaced by any function whose nth derivative never vanishes on $[0,1]$.

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